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NSWCDD/MP–99/17

MATHEMATICAL METHODS OF THREE–DIMENSIONAL EYE ROTATIONS

BASED UPON SPACECRAFT DYNAMICS NOTATION

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FEBRUARY 1999

REPORT DOCUMENTATION PAGE

Form Approved
OBM No. 0704–0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	February 99	Final

4. TITLE AND SUBTITLE

Mathematical Methods of Three–Dimensional Eye Rotations Based Upon Spacecraft Dynamics Notation

5. FUNDING NUMBERS

6. AUTHOR(s)

Kee Soon Chun

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)

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Dahlgren, VA 22448–5100

8. PERFORMING ORGANIZATION REPORT NUMBER

NSWCDD/MP–99/17

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

This report shows tutorial derivations (a form of student's guide) of all key equations needed in the research of three–dimensional eye rotations, which happens to coincide with virtually all key equations needed to understand the rotational motions of spacecrafts and missiles. Considering the fact that mathematics may not be the primary field of competence for most medical researchers, the author of this report took special care to explain the derivations from a very elementary level and to gradually progress to advanced equations in a logical, self–containing way. The author introduces some innovative ways of explaining otherwise difficult concepts and derivations in several topics.

Although this report starts from a modest level, it advances toward the end to fairly advanced esoteric equations, which even most aerospace engineers may not be aware of, or may not encounter during their career. These equations are highly desirable for a deeper insight into angular rotations. For this reason, although this report is written primarily with medical researchers in mind, it is equally recommended for scientists and engineers in the aerospace field dealing with the rotational dynamics of missiles, spacecraft, and aircraft in the Department of the Navy.

14. SUBJECT TERMS

Eye Rotations, Eye Movements, Rotational Dynamics, Vestibule–Ocular Reflex, Direction Cosine Matrix, Rotational Dynamics, Rotation Matrix, Coordinate Transmation Matrix, Euler's Theorem, Euler's Angles, Listing's Law, Donders' Law, Theorem of Coriolis, Spacecraft Dynamics, Missile/Rocket Motion.

15. NUMBER OF PAGES

228

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT5	18. SECURITY CLASSIFICATION OF THIS PAGE6	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

FOREWORD

This report shows tutorial derivations (a form of student's guide) of all key equations needed in the research of three–dimensional eye rotations, which happens to coincide with virtually all key equations needed to understand the rotational motions of spacecrafts and missiles. Considering the fact that mathematics may not be the primary field of competence for most medical researchers, the author of this report took special care to explain the derivations from a very elementary level and to gradually progress to advanced equations in a logical, self–containing way. The author introduces some innovative ways of explaining otherwise difficult concepts and derivations in several topics.

A few years back, the author was a visiting scientist for a year at the Department of Neurology of the Johns Hopkins School of Medicine at the invitation of his former advisor, Professor David A. Robinson. During his stay at Hopkins, he realized, based on requests and encouragement from colleagues, a need for a comprehensive book of this nature containing all key equations and their derivations in one volume. In writing this report, the author followed the pedagogical philosophy that the best way to understand the equations and to be able to use them with confidence is to be able to derive them from the basics in an easy–to–follow, yet rigorous way. For this reason, this report starts with elementary trigonometry by design, and advances logically to a higher level.

As the bibliography provided at the end of this report reveals, all equations in the report originated from applications to spacecraft and missile dynamics. The only deviation in this report is that the Head frame is identified with the reference frame, while the Eye frame is identified with the moving or rotating frame.

Although this report starts from a modest level, it advances toward the end to fairly advanced esoteric equations, which even most aerospace engineers may not be aware of, or may not encounter during their career. These equations are highly desirable for a deeper insight into angular rotations. For this reason, although this report is written primarily with medical researchers in mind, it is equally recommended for scientists and engineers in the aerospace field dealing with the rotational dynamics of missiles, spacecraft, and aircraft in the Department of the Navy.

Finally, the author of this report would like to dedicate this modest work of his to the benefit of medical researchers and students in the United States and abroad who may be struggling with unfamiliar mathematics in their work. This report is not copyrighted so that it may be reproduced freely for educational purposes.

Questions or comments concerning the contents of this document should be addressed to NSWCDD, Attn: Dr. Kee Soon Chun, K44, Dahlgren, Virginia 22448–5100, or telefaxed to 540–653–8382, or e–mail to <kChun@nswc.navy.mil>.

Approved by:

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PREFACE

Dr. Kee Soon Chun was first exposed to the oculomotor system around 1976 as a graduate student in my laboratory in the Wilmer Eye Institute at the Johns Hopkins School of Medicine where he did his thesis research for a doctorate in Electrical Engineering. We made a model of the vestibule–ocular reflex with a quick–phase mechanism to generate nystagmus (A model of Quick Phase Generation in Vestibule–Ocular Reflex, by K. S. Chun and D. A. Robinson, Biological Cybernetics 28, pp. 209–221(1978)). It introduced the concept of two functions of time: (1) The "when" curve which, when the eye position reached it, initiated a quick phase; and (2) the "where" curve to which the quick phase carried the eye. These are concepts that have been found useful subsequently.

Many years later, in 1996, Kee Soon was granted a sabbatical by the Navy. After working for many years on the mathematics underlying the flight of intercontinental ballistic missiles, he was motivated by a desire to turn his talents to something of a more humanitarian nature and thought again of Johns Hopkins and our laboratory. He joined us once again and helped our work on a neural network model of the neural integrator (the one that converts vestibular velocity commands to oculomotor position signals). While doing so, he became aware that much of the research in our laboratory (that of Dr. David Zee) involved the analysis of eye movements in all three–dimensions. This subject has been growing in importance over the last fifteen years, when one–dimensional analyses were felt to be largely understood, and it was realized how easy it was to measure torsion movements with the eye–coil/magnetic–field method. This has lead to the need for more sophisticated mathematics when dealing with such things as coordinate transformations, Listing's Law, and quaternions. All of this caught Kee Soon's fancy because it was not so very different from what one needs in studying the flight of a missile over a revolving planet. He was inspired to bring these mathematical tools together and catalogue them for the convenience of others. Thus, this publication came about.

The result might be regarded as a reference source; a place to go and remind oneself of the mathematical bases of the various tools we use in oculomotor analyses. For those not already comfortable with vectors and matrices, this text may be too terse and compact to serve as a learning tool although the sections on the Fick's and Helmholtz's coordinate systems may help the beginning student to understand these potentially confusing representations. Also the description of semicircular dynamics (Appendix B) is a useful introduction to anyone beginning a study of the vestibule–ocular reflex.

David A. Robinson, Professor
School of Medicine
Johns Hopkins University

January 1999

SECTION 1

INTRODUCTION

1.1 THREE–DIMENSIONAL EYE ROTATIONS

This report explains the basic mathematics necessary to understand three–dimensional (3D) eye rotations, in the easiest and the simplest way it can be presented.

The mathematical background assumed is not more demanding than the college freshman level.

The goal of this report is to show easy–to–understand derivations for the key equations used in the mathematics for 3D eye rotations. This is based on the philosophy that the ideal way to understand and apply the equations is to be able to understand their derivations. To the author's best knowledge at this writing, there seems to be no report or book of this nature available. This writing is a modest endeavor toward that direction, hoping someday someone will take over and continue the task.

Analysis of 3D eye rotations requires a lot of mathematics, and the particular mathematical methods that are commonly used stem from many areas that are themselves a wide field. This report only touches on the standard methods used today. Many other methods are used and can be used to gear the analysis to larger movements (such as the head movements). The reader must keep in mind that every method has its limitations for eye movement analysis and should be used cautiously.

Section 1 introduces virtually all the basic linear algebra, along with Appendix A, required to understand this report. Section 2 describes and demonstrates **Euler's Theorem**, which is a central concept in studying 3D rotations. Section 3 describes a geometrical approach to understanding **Listing's and Donders' Laws**. These laws are immediate applications of this theorem in eye movement analyses.

Section 4 covers **Euler Angles**. These are the minimum numbers of parameters required to describe a rotation in space. Two sets of Euler Angles used most commonly in eye movement applications, namely Fick's and Helmholtz's coordinate systems, are described in this section.

Central to understanding an application of the 3D rotations is the concept of **Rotation Matrices** which is covered in Section 4. For a person dealing with 3D rotations, it is important to understand rotation matrices, how to generate them within an experimental setup, and what they represent. Sections 5 through 13 include the mathematical and geometrical descriptions of these matrices, their characteristics, and their relationships to Fick's and Helmholtz's coordinates.

Sections 14 and 15 establish the concept of angular velocity in 3D rotations by demonstrating the **Theorem of Coriolis**.

Section 16 and the remaining sections turn our focus to another method of describing 3D rotations. As a result of **Euler's Theorem**, instead of describing a rotation in space as three consecutive rotations (resulting in Euler angles) we can define a set of parameters which defines an equivalent single–axis of rotation and the angle of rotation around this axis. Quaternions and rotation vectors are the two most widely used of such methods for describing rotations. These methods and their special mathematics are described in Sections 16 through 26.

1.2 REVIEW OF SOME ALGEBRA

To start, we review the **Pythagorean Theorem**. Referring to Figure 1-1, we see that $\cos\theta = x/r$ where r is the magnitude of the vector **r**, or

$$x = r\cos\theta \tag{1-1}$$

which is the component of **r** along the X–axis.

Also, we see that $\sin\theta = y/r$, or

$$y = r\sin\theta = r\cos\,[90^\circ - \theta] \tag{1-2}$$

which is the component of **r** along the Y–axis.

The method of determining the components of a vector (such as **r** in Figure 1-1) along the coordinate axes this way is identical to the method for determining the elements of a **Rotation Matrix**, or Coordinate Transformation Matrix (see Section 5, the Rotation Matrix).

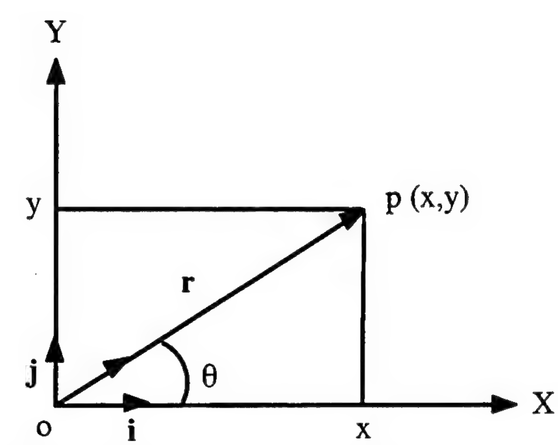


Figure 1-1. Example of Pythagorean Theorem

From Equation (1-1) and Equation (1-2):

$$x^2 + y^2 = [r\cos\theta]^2 + [r\sin\theta]^2$$

$$= r^2\,[\cos^2\theta + \sin^2\theta]$$

$$= r^2 \quad (\text{since } \cos^2\theta + \sin^2\theta = 1) \tag{1-3}$$

which is the familiar **Pythagorean Theorem**, and validates indirectly the method used to determine the x and y components of **r**. Denoting the unit vector along the X–axis by **i**, and along the Y–axis by **j**, we can represent the vector **r** in terms of x and y by:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y \quad \text{or} \quad \mathbf{r} = \mathbf{j}y + \mathbf{i}x \tag{1-4}$$

Note in Equation (1-4) that the vector addition commutes. We can do this because (referring to Figure 1-1) we can reach the point p (x, y) either by moving to the right first and then upward, or by moving upward first and then to the right. We emphasize this because the consecutive angular motions generally do not commute. This will be demonstrated later.

1.3 DOT PRODUCT

Denote two vectors **A** and **B** in a 3D orthogonal frame by

$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{1-5}$$

$$\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \tag{1-6}$$

where **i, j,** and **k** are the unit vectors along the x, y, and z axes respectively.

The **Dot Product** (also called **Scalar Product** or **Inner Product**) denoted by $\mathbf{A}\cdot\mathbf{B}$ (pronounced as A dot B) is defined to be

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta \tag{1-7}$$

where θ is the angle between **A** and **B**. Noting that $|\mathbf{i}| = |\mathbf{j}| = 1$ and $\mathbf{i}\perp\mathbf{j}$, we have using Equation (1-7):

$$\mathbf{i}\cdot\mathbf{j} = \cos\frac{\pi}{2} = 0 \tag{1-8}$$

Similarly,

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \cos\frac{\pi}{2} = 0 \qquad (1\text{-}9)$$

$$\mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \cos\frac{\pi}{2} = 0 \qquad (1\text{-}10)$$

also,

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = \cos 0 = 1 \qquad (1\text{-}11)$$

Therefore,

$$\mathbf{A} \cdot \mathbf{B} = [a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}] \cdot [b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}] \qquad (1\text{-}12)$$

reduces to, using Equations (1-9), (1-10), and (1-11):

$$\mathbf{A} \cdot \mathbf{B} = a_xb_x + a_yb_y + a_zb_z = \mathbf{B} \cdot \mathbf{A} \qquad (1\text{-}13)$$

The result of Equation (1-13) is a scalar value describing the projection of vector **A** onto vector **B** (or vice versa). If **A** and **B** are orthogonal to each other ($\theta=90^0$), then $\mathbf{A} \cdot \mathbf{B} = 0$.

Expressing **A** and **B** by column vectors,

$$\mathbf{A} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \qquad (1\text{-}14)$$

we may express $\mathbf{A} \cdot \mathbf{B}$ equivalently by

$$\mathbf{A}^T\mathbf{B} = [a_x\ a_y\ a_z]\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= a_xb_x + a_yb_y + a_zb_z \qquad (1\text{-}15)$$

The **transpose** of **A**, denoted by $\mathbf{A}^T$, is obtained by converting the columns of **A** into the rows of **A** one at a time in sequence.

Equation (1-15) above yields the same results as $\mathbf{A} \cdot \mathbf{B}$ given in Equation (1-13). $\mathbf{A}^T\mathbf{B}$ is referred to as the **Inner Product**, while $\mathbf{A} \cdot \mathbf{B}$ is referred to as the **Dot Product** although they both mean the same thing. It is also called **Scalar Product** because the results are scalar, not vector.

1.4 CROSS PRODUCT

The **Cross Product** (also called **Vector Product**) of two vectors **A** and **B** denoted by $\mathbf{A} \times \mathbf{B}$ (pronounced as A cross B) is another vector, defined by:

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta\, \underline{\mathbf{u}} \tag{1-16}$$

where θ is the angle between vectors **A** and **B**, and **u** is a unit vector perpendicular to the plane containing both **A** and **B**, and the direction of **u** is along the right hand thumb if its fingers curl from **A** to **B**. Hence, this convention is called right hand rule.

Note the result of Equation (1-16) is vector orthogonal to both vectors **A** and **B**. If $\mathbf{A} \times \mathbf{B} = 0$, then it means the vectors **A** and **B** lie along the same line implying θ must be 0.

It follows, using Equation (1-16):

$$\mathbf{i} \times \mathbf{j} = \sin\frac{\pi}{2}\mathbf{k} = \mathbf{k}; \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \sin\frac{\pi}{2}\mathbf{i} = \mathbf{i}; \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \sin\frac{\pi}{2}\mathbf{j} = \mathbf{j}; \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \tag{1-17}$$

Note

$$\mathbf{i} \times \mathbf{i} = \sin(0)\mathbf{k} = \mathbf{0} \text{ [zero vector]}. \tag{1-18}$$

Similarly,

$$\mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{i} \times \mathbf{i} = \mathbf{0} \text{ [zero vector]}. \tag{1-19}$$

It follows using Equation (1-5) and Equation (1-6):

$$\mathbf{A} \times \mathbf{B} = [a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}] \times [b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}] \tag{1-20}$$

which reduces to, using Equations (1-16) to (1-19):

$$\mathbf{A} \times \mathbf{B} = (a_yb_z - a_zb_y)\mathbf{i} + (a_zb_x - a_xb_z)\mathbf{j} + (a_xb_y - a_yb_x)\mathbf{k} \tag{1-21}$$

Now, the following determinant:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1-22}$$

gives the same results as Equation (1-21), and is much easier to compute. Therefore, $\mathbf{A} \times \mathbf{B}$ is conventionally expressed by:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1-23}$$

and can be treated as the definition of the **Cross Product** given in Equation (1-16).

1.5 MATRIX REPRESENTATION OF THE CROSS PRODUCT

For $\mathbf{R} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ and $\mathbf{W} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$, we get, using Equation (1-21) and Equation (1-23):

$$\begin{aligned} \mathbf{W} \times \mathbf{R} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_x & w_y & w_z \\ r_x & r_y & r_z \end{vmatrix} \\ &= [w_y r_z - w_z r_y]\mathbf{i} \\ &\quad + [w_z r_x - w_x r_z]\mathbf{j} \\ &\quad + [w_x r_y - w_y r_x]\mathbf{k} \end{aligned} \tag{1-24}$$

Now represent the vector $\mathbf{R} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ by an equivalent Column Matrix

$$\mathbf{R} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix},$$

and a vector $\mathbf{W} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$ by

$$\mathbf{W} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

We want to find a matrix representation $W^*\mathbf{R}$ corresponding to the vector representation $\mathbf{W} \times \mathbf{R}$ given in Equation (1-24) such that the components of $W^*\mathbf{R}$ is equal to the components of $\mathbf{W} \times \mathbf{R}$.

It turns out that if we express W^* by:

$$W^* = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \tag{1-25}$$

and perform the operation W^*R, we have:

$$\begin{aligned} W^*\mathbf{R} &= \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \\ &= \begin{bmatrix} w_y r_z - w_z r_y \\ w_z r_x - w_x r_z \\ w_x r_y - w_y r_x \end{bmatrix} \end{aligned} \tag{1-26}$$

which is, component–wise, equal to Equation (1-24).

It follows that the vector **Cross Product** $\mathbf{W} \times \mathbf{R}$ may be equivalently expressed by the **Matrix Product** of $W^*\mathbf{R}$ in the sense that its components are the same. This is convenient because, while $\mathbf{W} \times \mathbf{R}$ gives better physical insight, $W^*\mathbf{R}$ is much easier in computer applications.

If we take the transpose of Equation (1-25):

$$\begin{aligned} \left[W^*\right]^T &= \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & w_z & -w_y \\ -w_z & 0 & w_x \\ w_y & -w_x & 0 \end{bmatrix} \\ &= -\begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \\ &= -W^* \end{aligned} \tag{1-27}$$

We find that matrix W^* is **skew–symmetric** because

$$[W^*]^T = -W^* \quad \text{or} \quad W^* = -[W^*]^T. \tag{1-28}$$

See the following paragraph.

1.6 SOME DEFINITIONS OF MATRICES

The matrix obtained by interchanging the rows and columns of matrix A one by one is called the Transpose of A, and is denoted by A^T (A transpose).

For example, for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\left[A^T\right]^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

It follows that,

$$\left[A^T\right]^T = A \tag{1-29}$$

It can be easily shown by direct substitutions (of, say, three–by–three matrices) that,

$$(A + B)^T = A^T + B^T \tag{1-30}$$

$$(kA)^T = kA^T \quad \text{(where k is a scalar.)} \tag{1-31}$$

$$(AB)^T = B^T A^T \quad \text{(see Appendix A, Equation (A-8))} \tag{1-32}$$

A Square Matrix such that,

$$A = A^T \tag{1-33}$$

is called a **Symmetric Matrix**. Thus, for a three–by–three Symmetric Matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = A^T \tag{1-34}$$

A **Square Matrix** B is the inverse of a **Square Matrix** A if

$$AB = BA = I \tag{1-35}$$

The inverse of A, if it exists, is denoted by A^{-1}. Thus using $B=A^{-1}$ in Equation (1-35),

$$AA^{-1} = A^{-1}A = I \tag{1-36}$$

where I is the **Identity Matrix**. For instance, the three–by–three Identity Matrix is given by:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1-37}$$

A Square Matrix A is called an **Orthogonal Matrix** if

$$AA^T = A^TA = I \tag{1-38}$$

Thus, for a three–by–three Orthogonal Matrix A:

$$\begin{aligned} AA^T &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= A^TA = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{1-39}$$

Comparing Equation (1–36) with Equation (1–38), we conclude the following for an Orthogonal Matrix A,

$$A^T = A^{-1} \tag{1-40}$$

It is demonstrated, in Appendix A, using two three–by–three Orthogonal Matrices A and B that,

$$[AB]^T = B^TA^T = B^{-1}A^{-1} = [AB]^{-1} \tag{1-41}$$

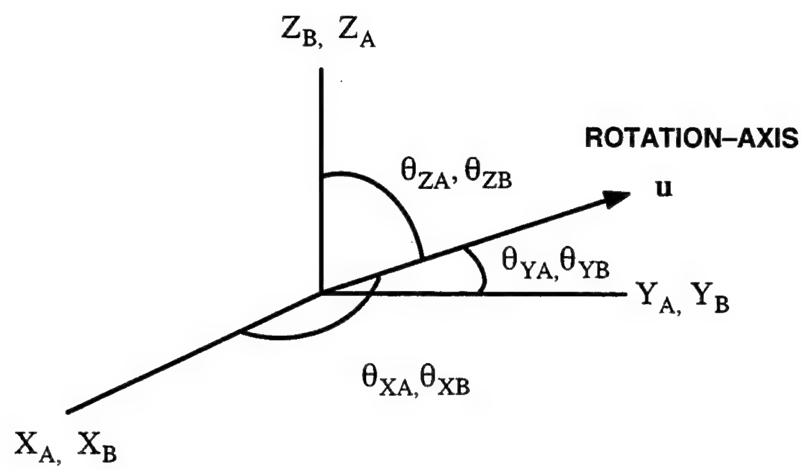
SECTION 2

EULER'S THEOREM OF A SINGLE, EQUIVALENT ROTATION

Euler's Theorem states that if a moving frame initially coincident with a fixed reference frame makes any number of rotations, regardless of how it reaches the final orientation, there always exists a single equivalent rotation with a finite angle about a single–axis through the origin.

In Section 3, we will describe a direct application of this theorem to eye rotation analysis.

To demonstrate the theorem, consider an orthogonal Frame B (with axes X_B, Y_B, and Z_B) rotating about an arbitrary–axis **u** fixed in space (called the rotation–axis) relative to another orthogonal Frame A (with axes X_A, Y_A and Z_A), which is fixed in space and serves as a reference frame. Frame B is initially coincident with Frame A, as shown in Figure 2-1.



INITIALLY FRAME B IS COINCIDENT WITH FRAME A

Figure 2-1. Example of Euler's Theorem

In Figure 2-1, **u** is an unit vector rigidly attached to the movable Frame B, while maintaining a fixed direction relative to Frame A, (but not attached to Frame A). The unit vector **u** makes angle θ_X with both X_A and X_B axes, angle θ_Y with both Y_A and Y_B axes, and angle θ_Z with both Z_A and Z_B axes initially as shown in Figure 2-1. Note that the angles θ_{XA}, θ_{YA}, and θ_{ZA} that **u** makes with Frame A remain constant, because the direction of **u** relative to Frame A is fixed. Also, the

angles θ_{XB}, θ_{YB}, and θ_{ZB} that **u** makes with Frame B remain constant, because Frame B is rotating around the axis of **u**, and **u** is rigidly attached to Frame B.

Now, rotate Frame B around **u** by an arbitrary angle ϕ while maintaining the direction of **u** fixed relative to Frame A (relative to space), as shown in Figure 2-2.

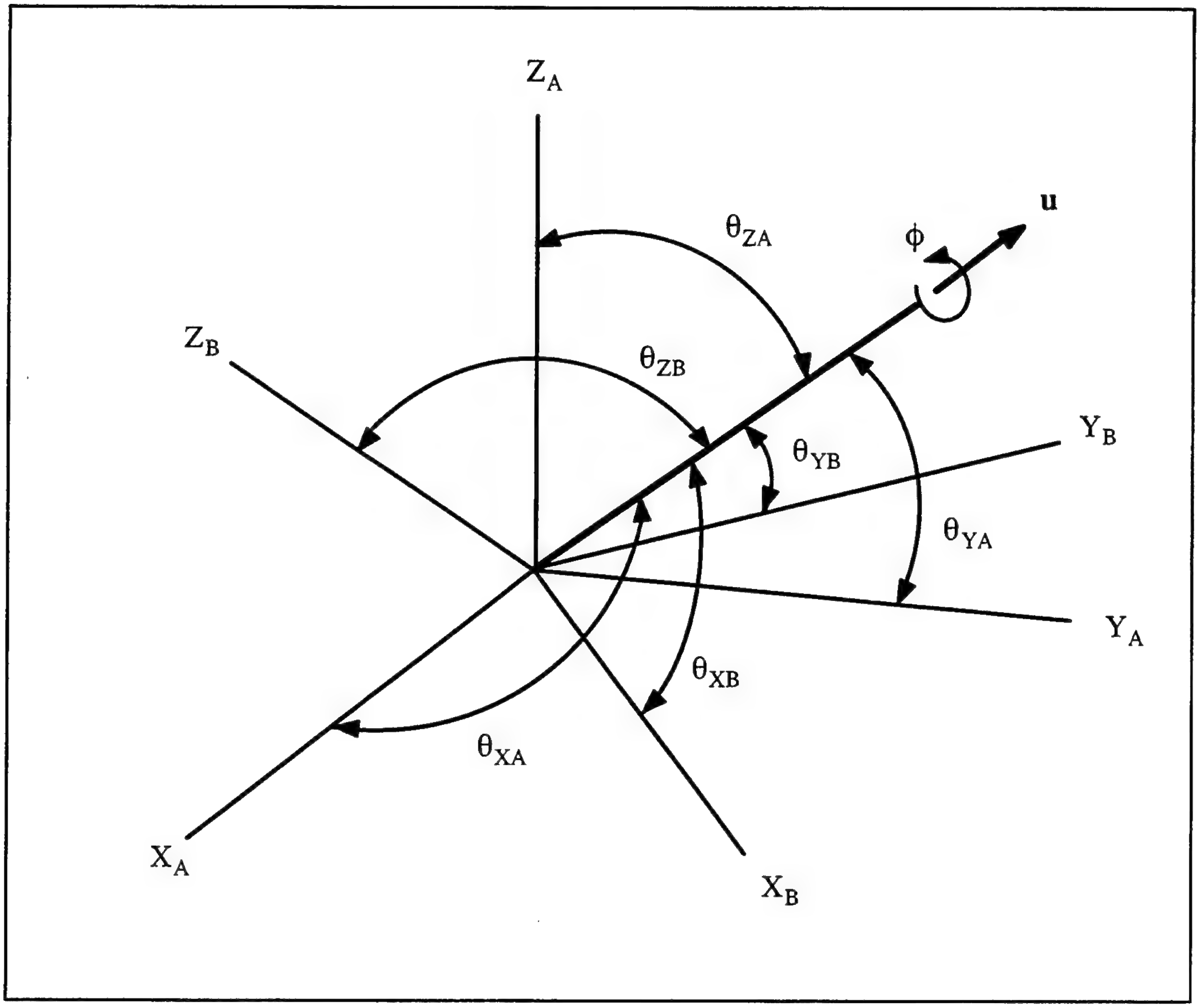


Figure 2-2. Frame B Rotated Around u

The vector **u** is located in the reference Frame A by the angles θ_{XA}, θ_{YA}, and θ_{ZA}, as shown in Figure 2-2. After the rotation, **u** is located in the rotating Frame B by the angles θ_{XB}, θ_{YB}, and θ_{ZB}. Because of Euler's theorem, and in reality,

$$\theta_{XA} = \theta_{XB}, \quad \theta_{YA} = \theta_{YB}, \quad \theta_{ZA} = \theta_{ZB} \tag{2-1}$$

We reached the orientation of Frame B in Figure 2-2 by an arbitrary rotation ϕ around a unit vector pointing to an arbitrary (but fixed) direction. This rotation does not change θ_{XB}, θ_{YB}, or θ_{ZB}, as previously explained.

The Frame B that was initially aligned with Frame A could have reached the final orientation of Frame B in Figure 2-2 by a sequence of many rotations about many different axes of rotation. But we are searching for a unique way to define the orientation of Frame B relative to Frame A. Establishing **u** by θ_X, θ_Y, and θ_Z and rotation angle ϕ produces this unique definition of the orientation of Frame B relative to Frame A.

Therefore, it is obvious that for any orientation of Frame B, there always exists a single, equivalent rotation–axis with an equivalent rotation angle that could move Frame B from its initial orientation in the reference Frame A to its final orientation. As an analogy, consider two points in a plane, such as P and Q. Since there are many paths that go from P to Q, we want to define a single path — the straight line — as the unique passage from P to Q.

Remember, unit vector **u** always makes the same orientation angles θ_X, θ_Y and θ_Z with respect to both Frame A and Frame B. Thus, it has the same components in both Frame A and Frame B. This is shown in the following, denoting the magnitude of **u** by u which is unity:

$$x_A = u\cos\theta_x = \cos\theta_x = x_B$$

$$y_A = u\cos\theta_y = \cos\theta_y = y_B$$

$$z_A = u\cos\theta_z = \cos\theta_z = z_B \qquad (2\text{-}2)$$

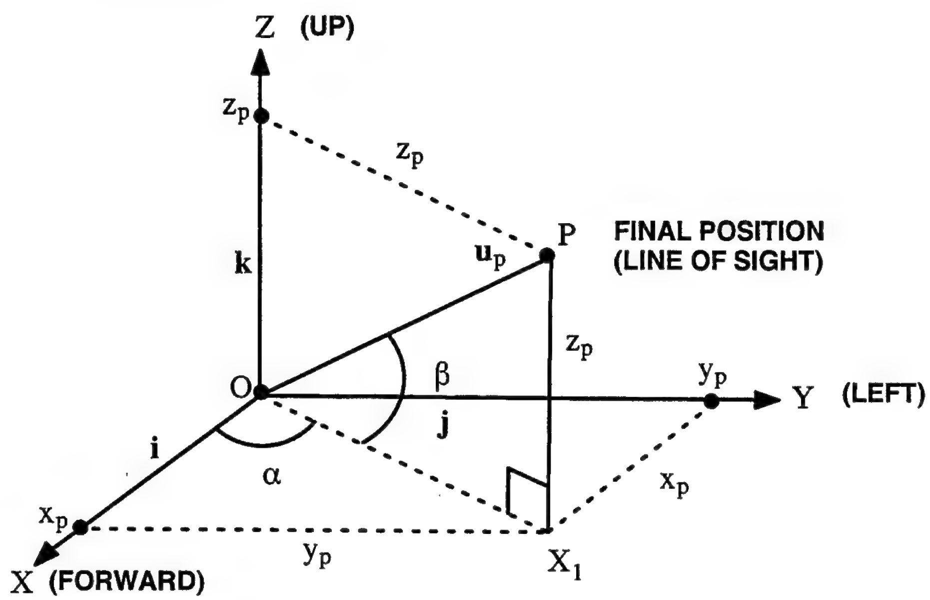
SECTION 3

LISTING'S LAW AND DONDERS' LAW BY SIMPLE GEOMETRY

Euler's Theorem of single equivalent rotation has an immediate application to eye movements expressed in **Donders' Law** and **Listing's Law**.

Donders' Law states that every eye position in space is described by a single orientation (fixed bearing and elevation – α and β in Figure 3-1) of the eye.

Listing's Law states that the eye can reach the final orientation through a single equivalent rotation about an axis of rotation. The axes of rotation for all final eye orientation will be in the same plane, called **Listing's Plane**.



INITIAL POSITION
(LINE OF SIGHT)

PRIMARY/REFERENCE
POSITION

Figure 3-1. The Orientation of Line of Sight

Referring to Figure 3-1, the line of sight of the initial eye position is OX along the X–axis, which we may consider as the primary/reference position; its final position is OP with the bearing

α and the elevation β. The eye may have reached the final position OP from the primary position by various sequences or paths.

Recall from Euler's Theorem that there exists an axis of rotation which brings OX to OP by a single equivalent rotation, regardless of how the final position OP may have been reached from the initial position OX. Listing's Law uses Euler's Theorem to find the Listing plane.

We now describe two methods for finding Euler's axis of single equivalent rotation.

In the first method, Euler's axis of rotation is found by constructing a perpendicular line, through origin, to the plane defined by lines formed by OX and OP. The perpendicular line is OZ′ in Figure 3-2 and OZ″ in Figure 3-3. The line is the Euler's rotation–axis that brings the OX to the OP by the right hand rule, with the thumb pointing in the direction of the perpendicular line, OZ′.

Denoting the unit vector along OX by $\mathbf{u}_X$ and the unit vector along OP by $\mathbf{u}_P$, the vector cross product $\mathbf{u}_X \times \mathbf{u}_P$ yields a vector $\mathbf{L}$ which is perpendicular to the plane formed by $\mathbf{u}_X$ and $\mathbf{u}_P$. That is,

$$\mathbf{L} = \mathbf{u}_X \times \mathbf{u}_P \tag{3-1}$$

Referring to Figure 3-2, expressing $\mathbf{u}_P$ by its components in the X–Y–Z frame by

$$\mathbf{u}_p = \mathbf{i}x_p + \mathbf{j}y_p + \mathbf{k}z_p \tag{3-2}$$

and realizing that $\mathbf{u}_X$ is equal to $\mathbf{i}$ since it is a unit vector along the X–axis, we have based on Equation (3-1), using Equations (1-17) and (1-18):

$$\begin{aligned}\mathbf{L} &= \mathbf{i} \times \left(\mathbf{i}x_p + \mathbf{j}y_p + \mathbf{k}z_p\right) \\ &= \mathbf{i} \times \mathbf{i}x_p + \mathbf{i} \times \mathbf{j}y_p + \mathbf{i} \times \mathbf{k}z_p \\ &= 0 + \mathbf{k}y_p - \mathbf{j}z_p \\ &= \mathbf{j}z_p + \mathbf{k}y_p \end{aligned} \tag{3-3}$$

Equation (3-3) shows that $\mathbf{L}$ must lie in the Y–Z plane since it has no $\mathbf{i}$ component. Since $\mathbf{u}_P$ may represent any final eye position, the Equation (3-3) shows that all axes of rotation corresponding to any final eye position must lie in the Y–Z plane, which is called **Listing's Plane**.

The Equation (3-3) suggests that the angle ϕ which $\mathbf{L}$ makes with the Z–axis (see Figure 3-2) may be computed from:

$$\tan\phi = \frac{|z_p|}{|y_p|} \quad \text{or} \quad \phi = \tan^{-1}\frac{|z_p|}{|y_p|} \tag{3-4}$$

Referring to Figure 3-1, if we know the bearing α and the elevation β of OP or $\mathbf{u_P}$, we may determine x_p, y_p, and z_p in Equation (3-2) by inspection as:

$$x_p = (\cos\beta)\cos\alpha$$

$$y_p = (\cos\beta)\sin\alpha$$

$$z_p = \sin\beta \qquad (3\text{-}5)$$

In the second method, Euler's rotation–axis is found by rotating coordinate frames in the following manner which shows that the angle ϕ is the torsion angle about the X–axis (primary/reference axis).

Step 1: Construct a plane containing OX and OP; find the intersection of this plane with the original Y–Z plane as shown in Figure 3-2. Call this intersection the Y′–axis. The line OY′ makes an angle ϕ with the OY–axis, or $\angle YOY'=\phi$.

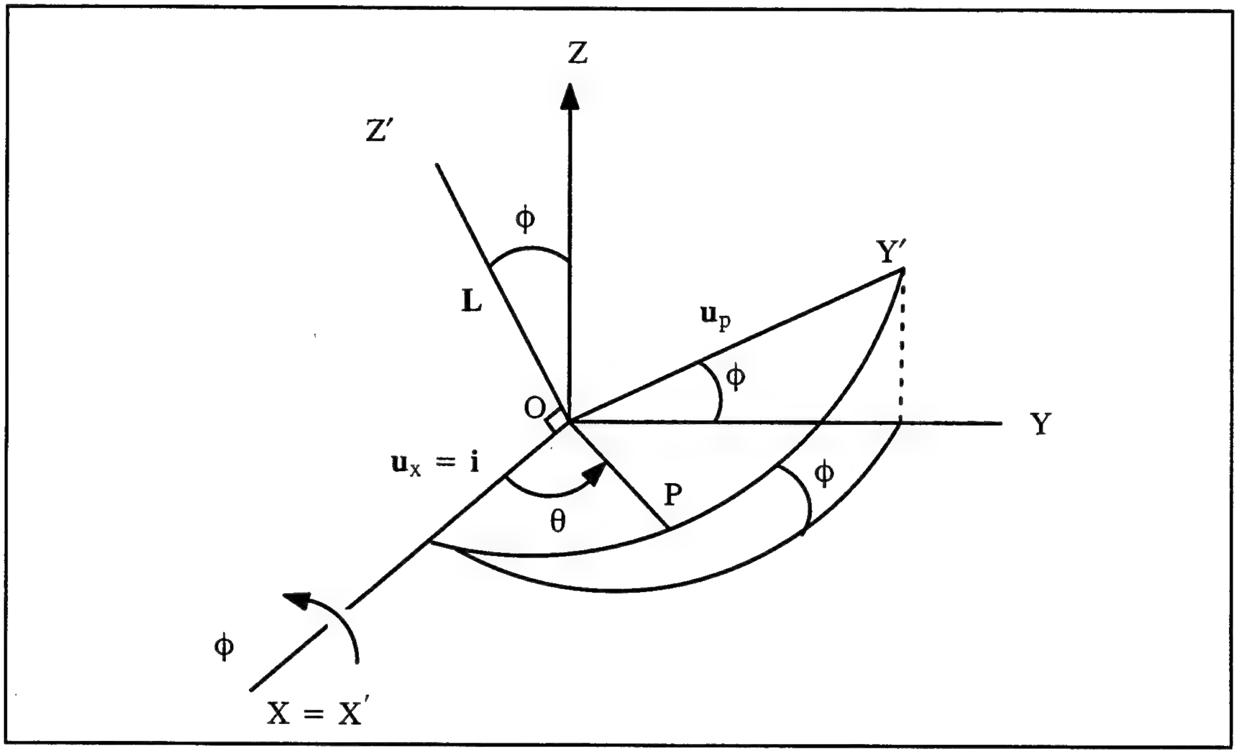


Figure 3-2. Rotation of X–Y–Z Frame about X–Axis (Primary/Reference Axis)

Step 2: Rotate the X–Y–Z frame about the X–axis until the Y–axis reaches the Y′–axis, or until the rotation angle about X–axis is equal to ϕ. Call the new frame X′–Y′–Z′.

Then X–axis coincides with X′–axis, while the Y–axis moves to Y′–axis through angle ϕ. The Z–axis moves to Z′–axis also with angle ϕ while staying on the original Y–Z plane.

Note that both Z′–axis and Y′–axis are perpendicular to the X–axis because they remain in the original Y–Z plane.

Step 3: Now rotate X′–Y′–Z′ frame about the Z′–axis by the angle θ, until OX (primary position) coincides with OP (final position) as shown in Figure 3-3. Call this new frame X″–Y″–Z″ frame.

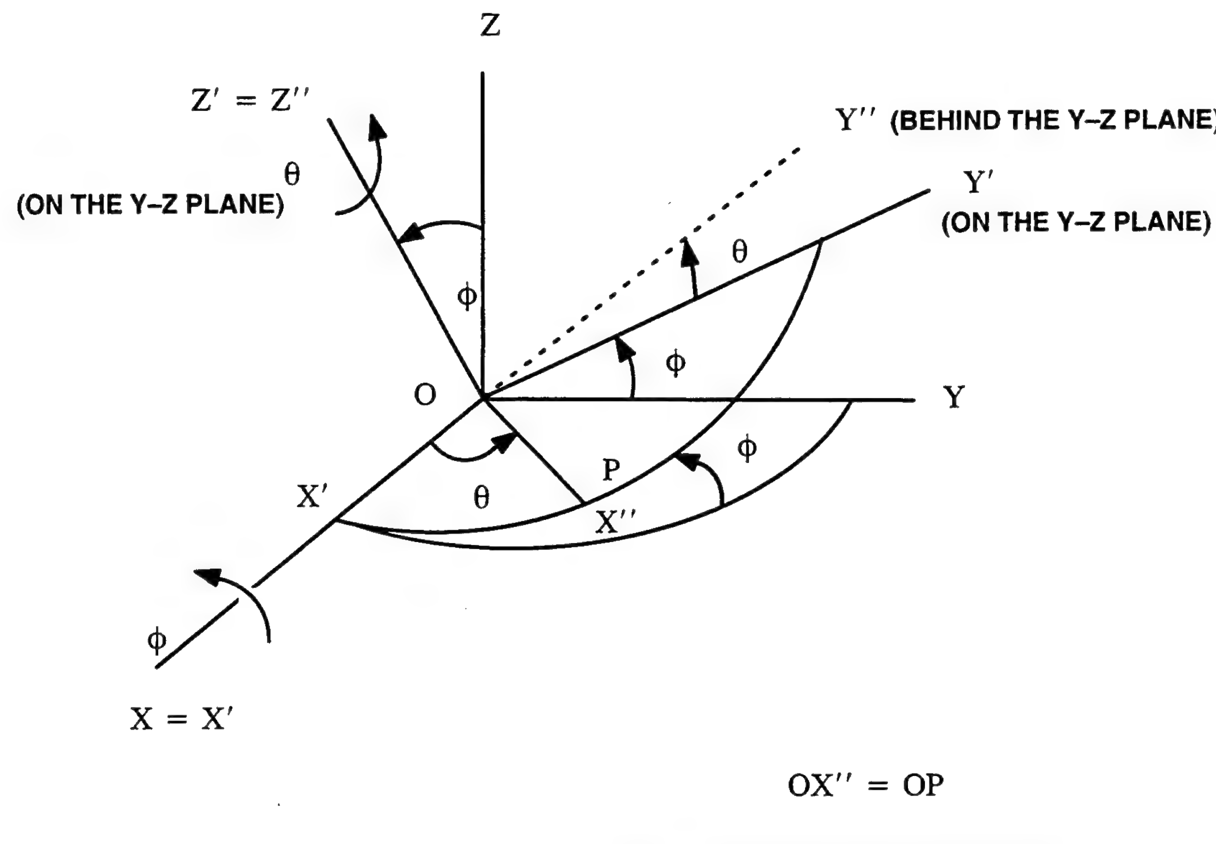


Figure 3-3. Euler's axis of Rotation on the Listing's Plane

Then the X–axis (or the X′–axis) moves to X″–axis, Y′–axis moves to Y″–axis, and Z′–axis remains at the same direction and becomes Z″–axis.

Note that the Z''–axis (Z'–axis) lies on the original Y–Z plane. The X''–axis lies on the X'–Y' plane (or X'–Y' plane). The Y''–axis is located behind the vertical Y–Z plane, while still lying on the X'–Y' plane. The angle θ is the single, equivalent rotation angle that makes the primary eye position OX to the final eye position OP or OX''.

To summarize, in order to move the eye from the primary eye position OX (X–axis) to the final eye position OP (X''–axis) by a single equivalent rotation, the eye has to be rotated about the Z''–axis in Listing's plane (which is the Y–Z plane) by angle θ (the single equivalent rotation). Regardless of any direction of the line OP, we can see that the axis of single equivalent rotation always lies on the unique plane called the **Listing's plane**.

We see that Donders' Law and Listing's Law imply that the angular rotation ϕ (in Figures 3-2 and 3-3), called **cyclotorsion** or simply **torsion**, is fixed for each final position (line of sight), regardless of the different paths the eye might have taken to reach there.

So, practically, we can test Listing Law by measuring torsion ϕ at any final eye position. The dot product of the corresponding axis of rotation with the primary/reference position should be equal to 0 (perpendicular).

In reality, the primary position is slightly lifted upward from the head's straight forward reference position. This is equivalent to the rotation of the initial eye position (which is coincident with the reference head position) by a small angle around the Y–axis.

For this situation, the Listing's plane is tilted backward by the same small angle, and its axis may be determined in the displaced eye frame by the similar method as explained in this section. Coordinates may be transformed back to the reference head frame by means of a Rotation Matrix for the Y–axis rotation. This will be discussed in Section 5.

SECTION 4

EULER ANGLES

Euler Angles orient one orthogonal Frame B displaced relative to another orthogonal Frame A by making three sequential rotations of Frame B relative to Frame A. The Frame B is initially aligned with the Frame A by a common origin. The first rotation is about any axis of the initial Frame B. The second rotation is about either of the two axes of the displaced Frame B not used for the first rotation. The third rotation is about either of the two axes of the Frame B not used for the second rotation. Thus, the number of permutations for the possible sequences of three Euler Angles is (3) (2) (2) or 12.

One particular sequence or set of Euler Angles has had wide application. In this set, the initial Frame B_0 with X_0, Y_0, and Z_0 axes is rotated about its Z_0–axis through an angle ϕ, resulting in the Frame B_1 with X_1, Y_1, and Z_1 axes (see Figure 4-1). Note that the Z_1–axis coincides with the Z_0–axis during the first rotation. In the second rotation, the Frame B_1 is rotated about its X_1–axis through an angle θ, resulting in the Frame B_2 with X_2, Y_2, and Z_2 axes (see Figure 4-2). Note that the X_2–axis coincides with the X_1–axis during the second rotation. In the third rotation, the Frame B_2 is rotated about its Z_2–axis (displaced Z_1–axis) by an angle ψ, resulting in the Frame B_3 with X_3, Y_3, and Z_3 axes (see Figure 4-3). Note that the Z_3–axis coincides with the Z_2–axis during the third rotation. This Z_0, X_1, Z_2 Euler Set is commonly used for robotic applications, and for aircraft and missile applications.

Out of twelve possible sets of Euler Angles, only two sets are historically used in eye rotation analyses: Fick's system and Helmholtz's system of Euler Angles. Fick's system will be described in Section 4.1, and Helmholtz's system will be described in Section 4.2.

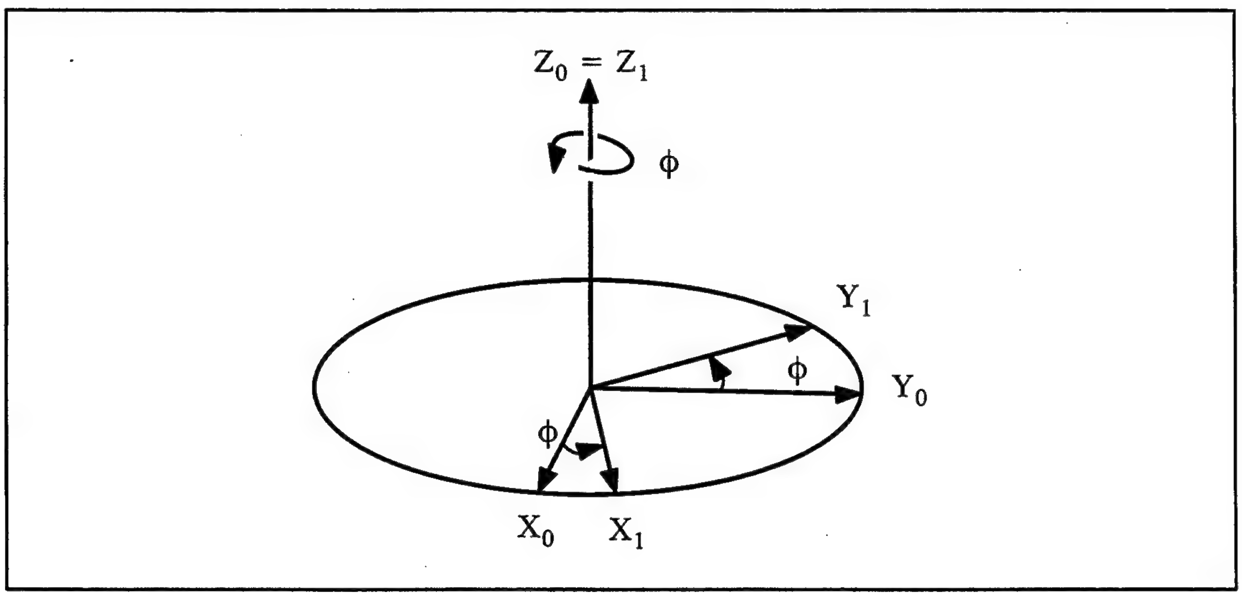


Figure 4-1. Frame B_1

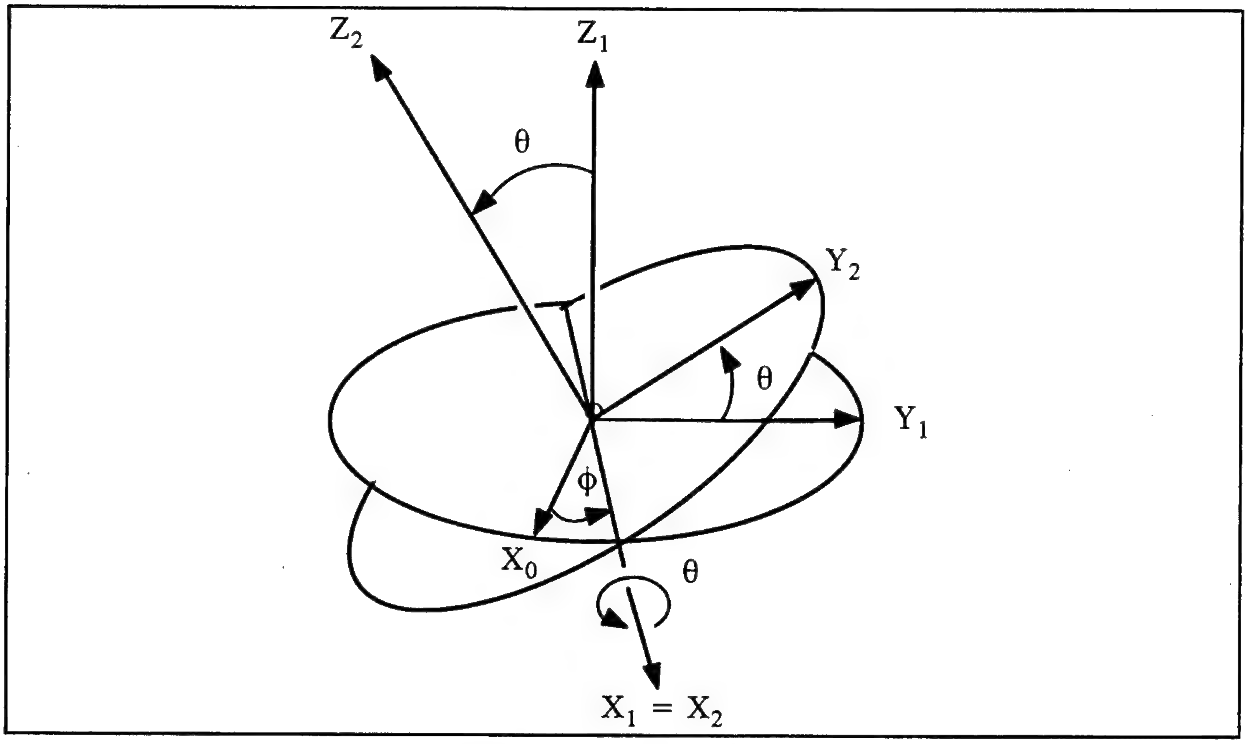


Figure 4-2. Frame B_2

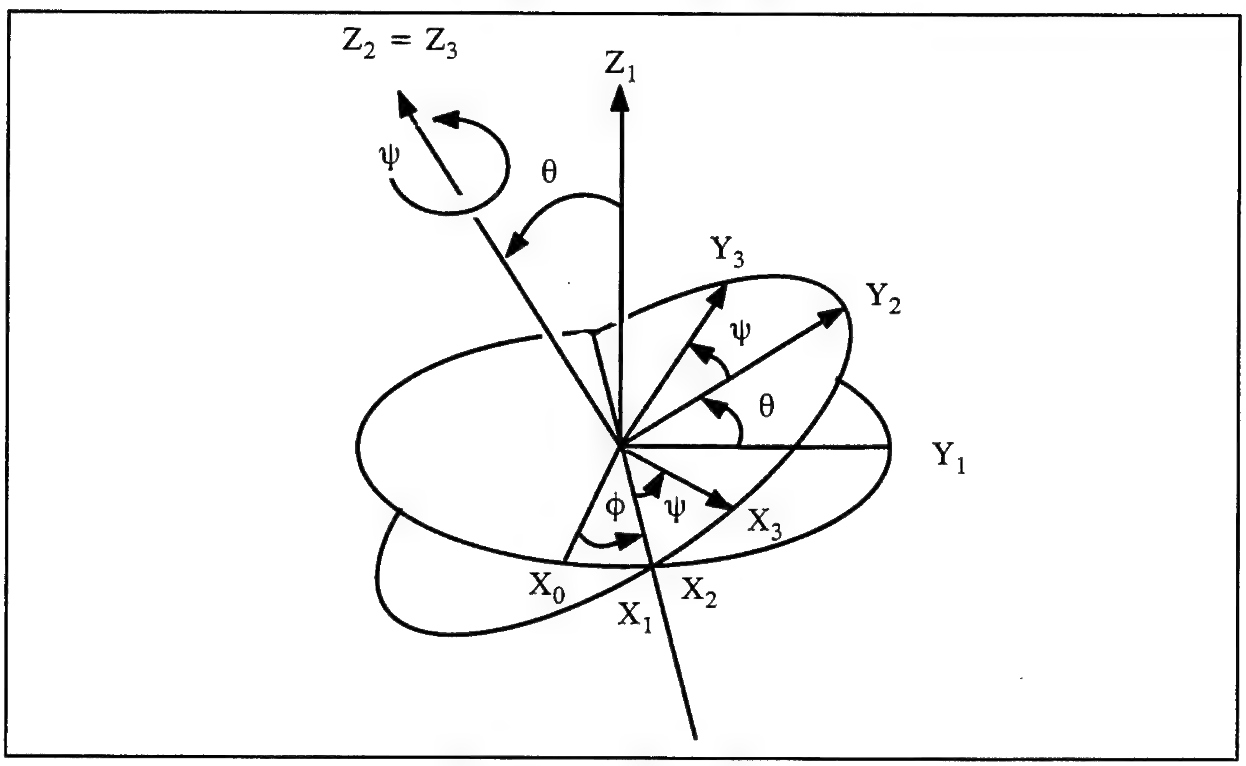


Figure 4-3. Frame B_3

4.1 FICK'S SYSTEM OF EULER ANGLES

In the Fick's system, as in the Helmholtz's system, initially both the Head frame and the Eye frame are coincidentally aligned with both of their X–axes pointed straight forward, Y–axes horizontally leftward, and Z–axes vertically upward.

The sequence of rotations in Fick's system is about the Z_0–axis of the initial Frame E_0, followed by rotation about the Y_1–axis of the Frame E_1 (the displaced Frame E_0) followed by rotation about the X_2–axis of the Frame E_2 (displaced Frame E_1). This is a Z_0, Y_1, X_2 Euler Set.

For the first rotation of Fick's system, the initial Frame E_0 (E for Eye) with its X_0, Y_0, Z_0 axes is rotated about the Z_0–axis resulting in the Frame E_1 with X_1, Y_1, Z_1 axes. This is shown in Figure 4-4 using the rotation angle of 90^0 for clarity and ease of comprehension. We, of course, understand that 90^0 eye rotation is not realizable physiologically. All eye rotation angles of Fick's system, as well as those of Helmholtz's system, are constrained to be well less than 90^0 in reality.

For the second rotation, the Frame E_1 is rotated about its Y_1–axis, resulting in the Frame E_2 with X_2, Y_2, Z_2 axes. This is shown in Figure 4-5 using the rotation angle of 90^0. Note that Y_1–axis is pointed into the paper. Thus, the thumb is pointed into the paper as well, when the right hand rule is applied (Figure 4-5).

For the third rotation, the Frame E_2 is rotated about its X_2–axis, resulting in the Frame E_3 with X_3, Y_3, Z_3 axes (Figure 4-6).

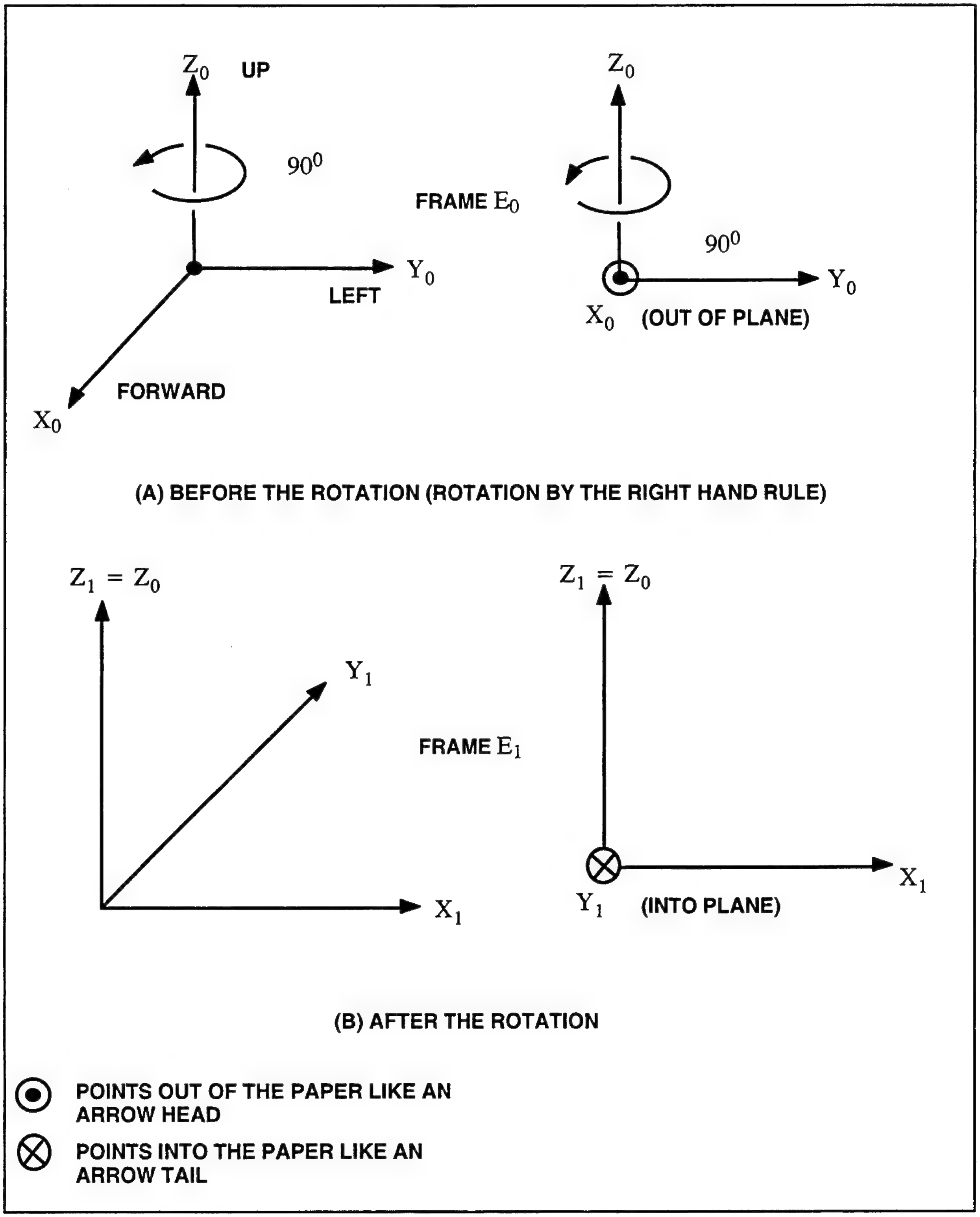


Figure 4-4. First Rotation About the Z_0–Axis in Fick's System

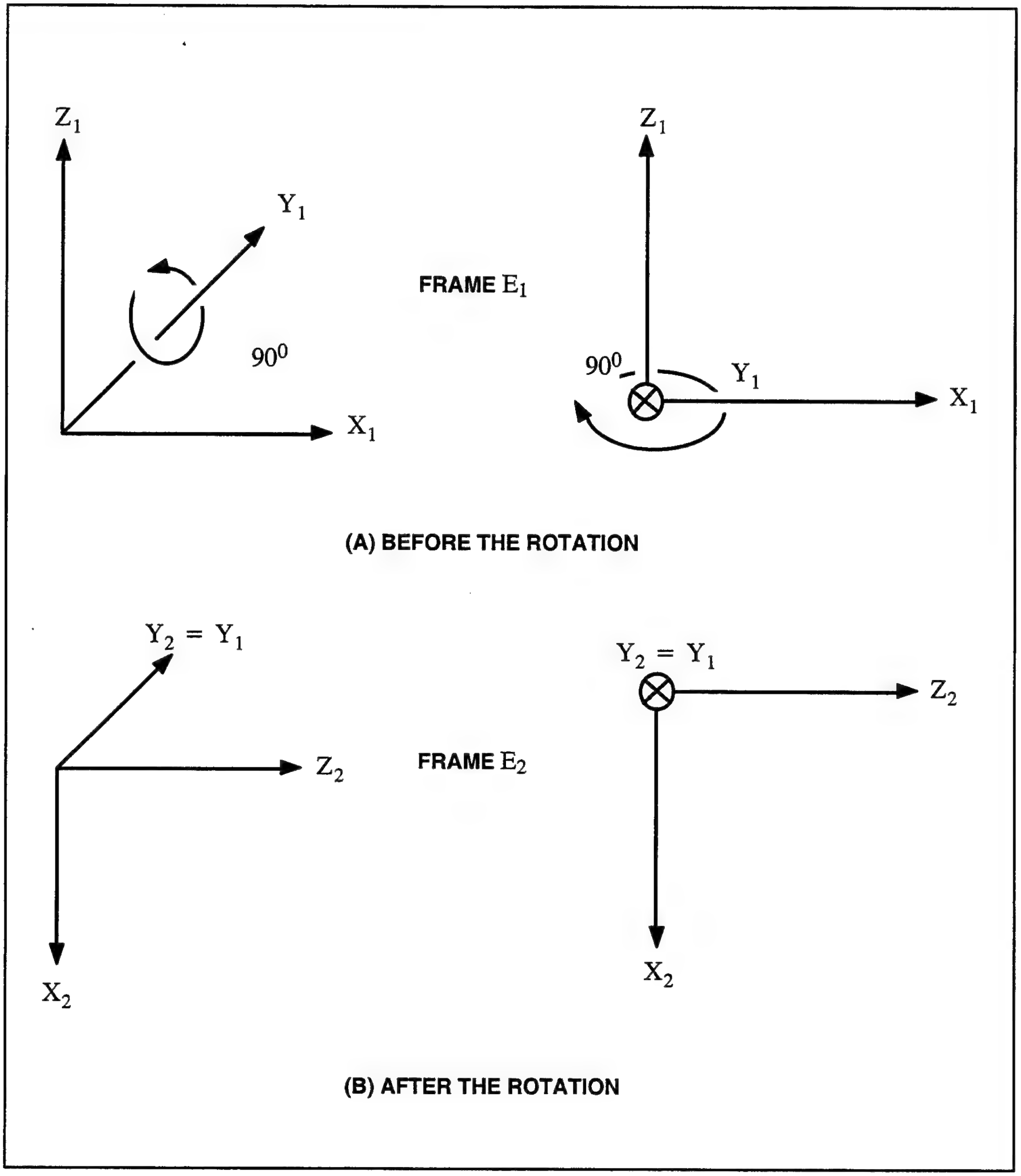


Figure 4-5. Second Rotation About the Y_1–Axis in Fick's System

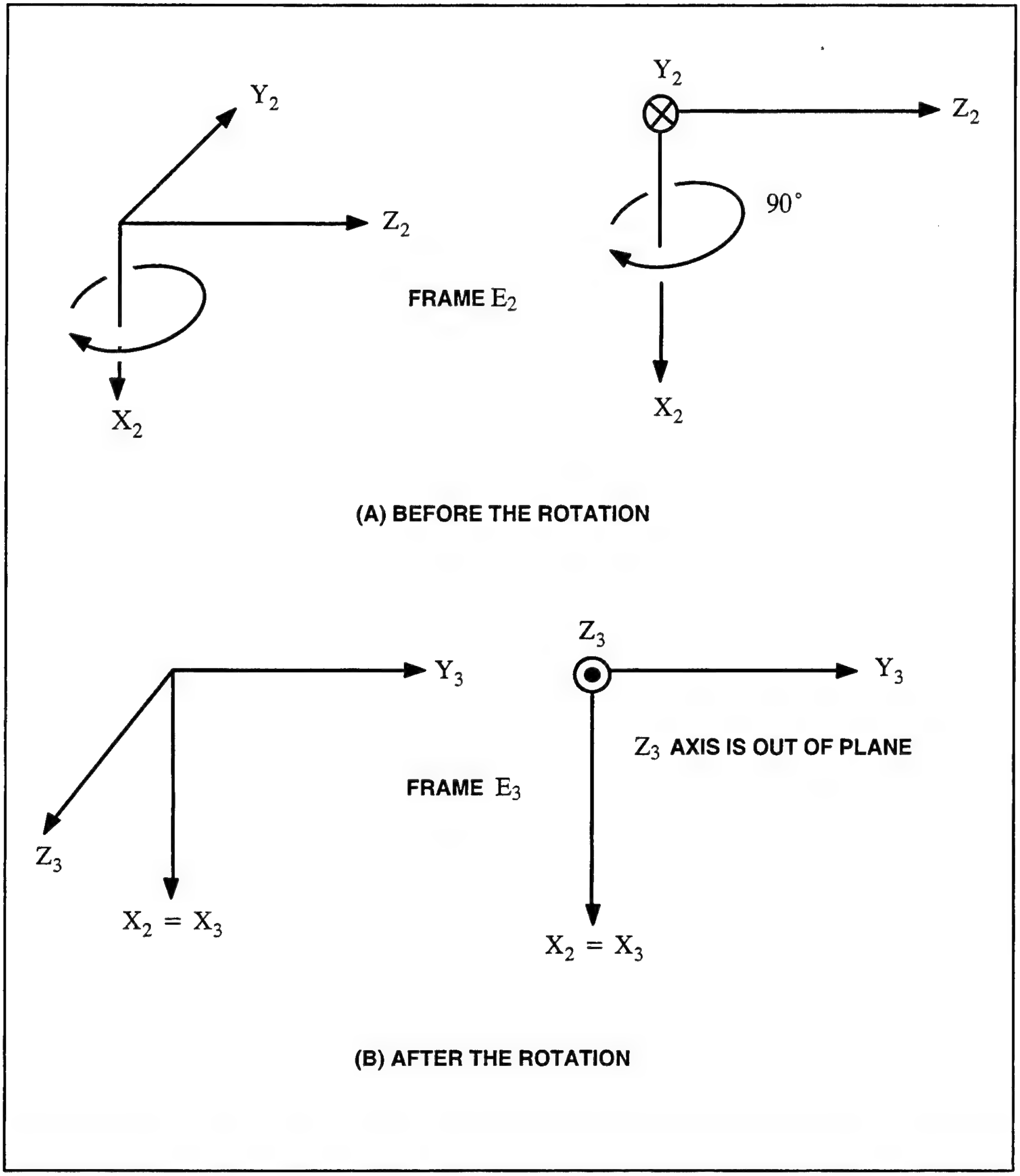


Figure 4-6. Third Rotation About the X_2–Axis in Fick's System

4.2 HELMHOLTZ'S SYSTEM OF EULER ANGLES

In the Helmholtz's system, as in the Fick's system, initially the Head frame coincides with the Eye frame with the X–axis pointed forward, the Y–axis leftward, and Z–axis upward.

The sequence of rotations in Helmholtz's system is first about the Y_0–axis of the initial Frame E_0 , followed by rotation about the Z_1–axis of the Frame E_1 (the displaced Frame E_0), followed by rotation about the X_2–axis of the Frame E_2 (the displaced Frame E_1). This is a Y_0, Z_1, X_2 Euler Set.

For the first rotation of Helmholtz's system, the initial Frame E_0 is rotated about its Y_0–axis, resulting in the Frame E_1 with X_1, Y_1, Z_1 axes. This is shown in Figure 4-7, using rotation angle of 90^0 again for demonstration only. All rotation angles of Helmholtz's system are constrained to be less than 90^0 in reality.

For the second rotations, the Frame E_1 is rotated about its Z_1–axis resulting in the Frame E_2 with X_2, Y_2, Z_2 axes. This is shown in Figure 4-8, using a rotation angle of 90^0.

For the third rotation, the Frame E_2 is rotated about its X_2–axis, resulting in the Frame E_3 with X_3, Y_3, Z_3 axes. This shown in Figure 4-9, using the rotation angle of 90^0.

Note that the first two rotations of Fick's system are the Z_0–axis rotation followed by the Y_1–axis rotation, while those of Helmholtz's system are the Y_0–axis rotation followed by the Z_1–axis rotation, which is the reverse order of the former. In both systems, the third rotation is about the X_2–axis.

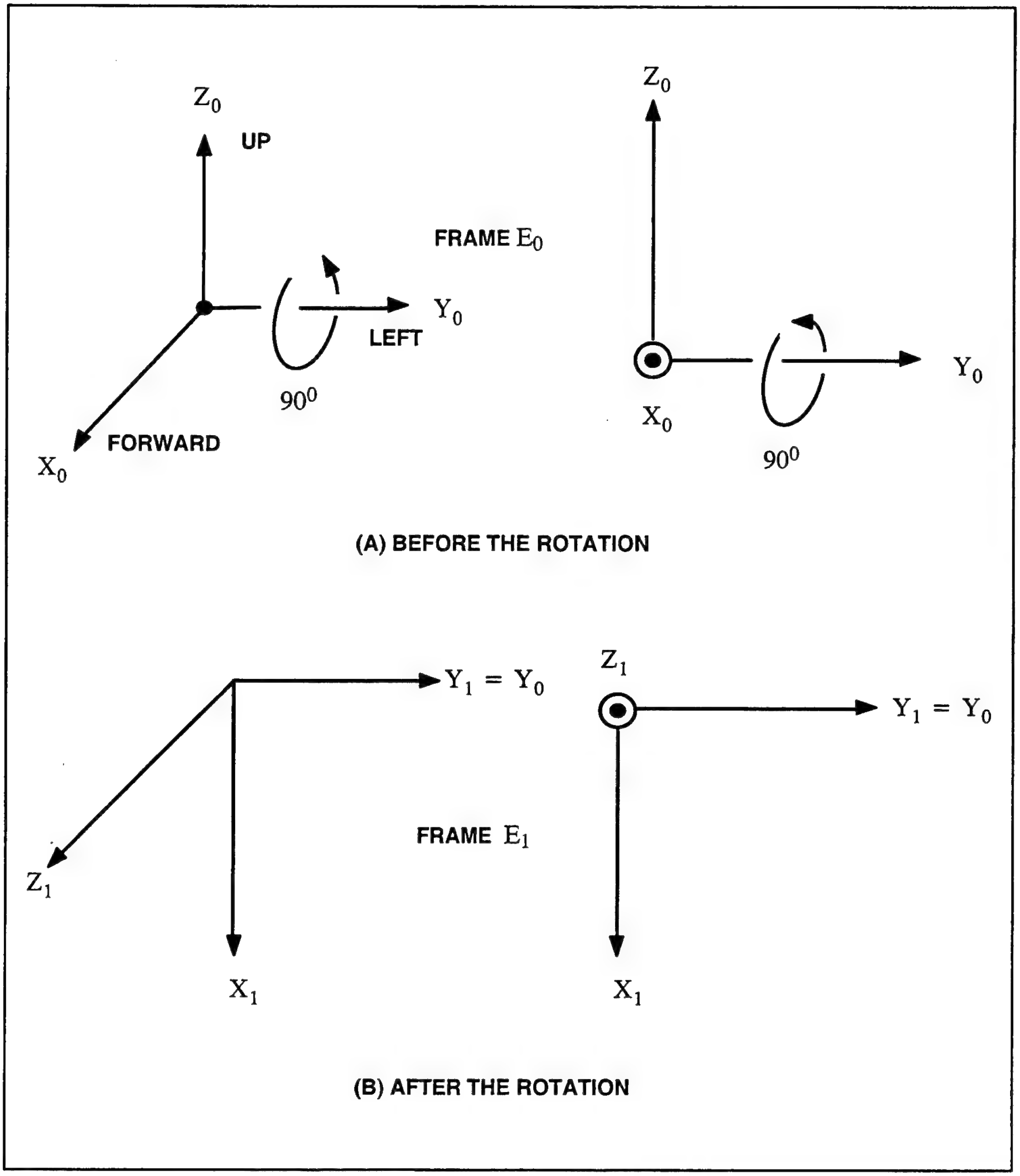


Figure 4-7. First Rotation About the Y_0–Axis in Helmholtz's System

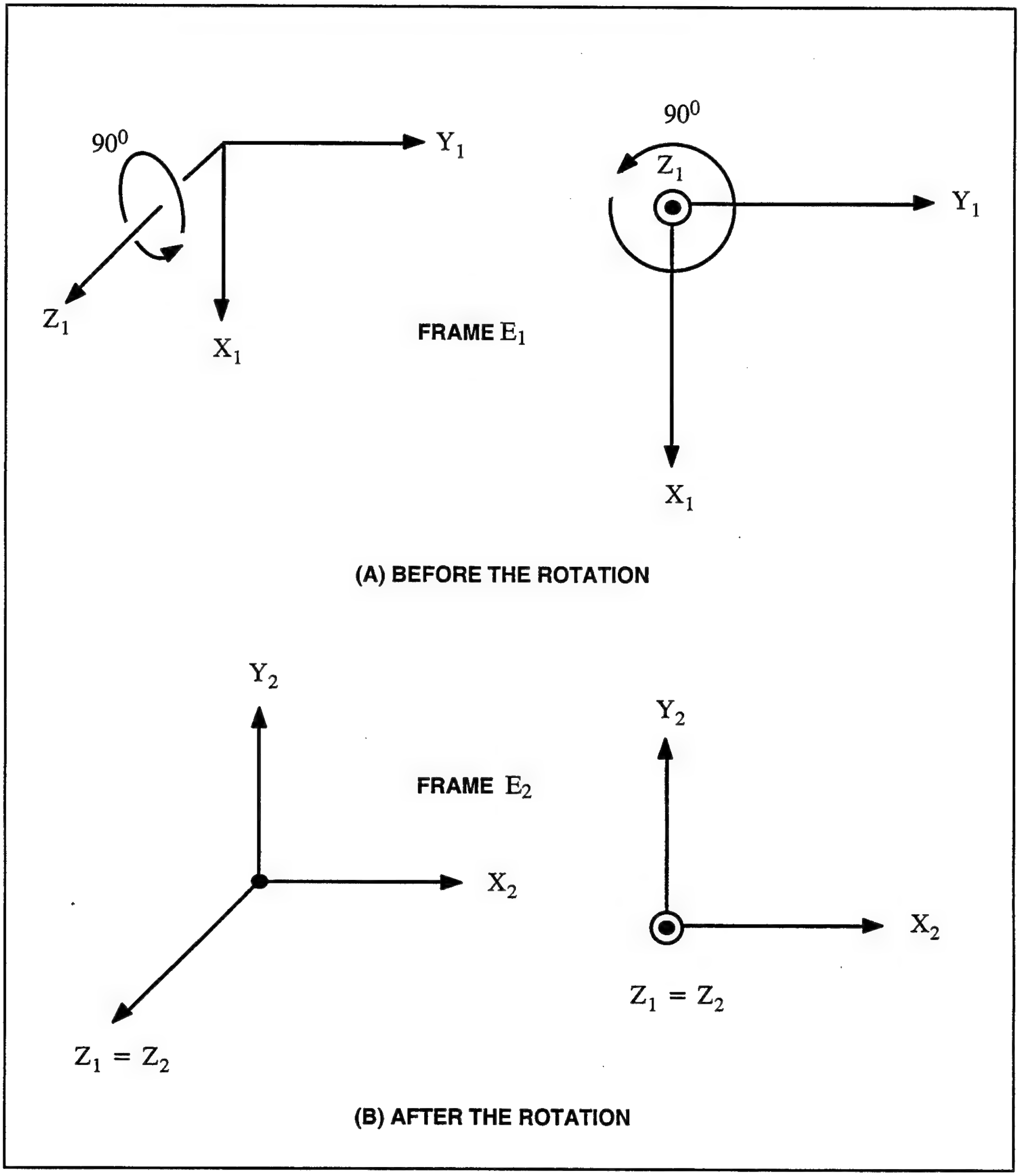


Figure 4-8. Second Rotation About the Z_1–Axis in Helmholtz's System

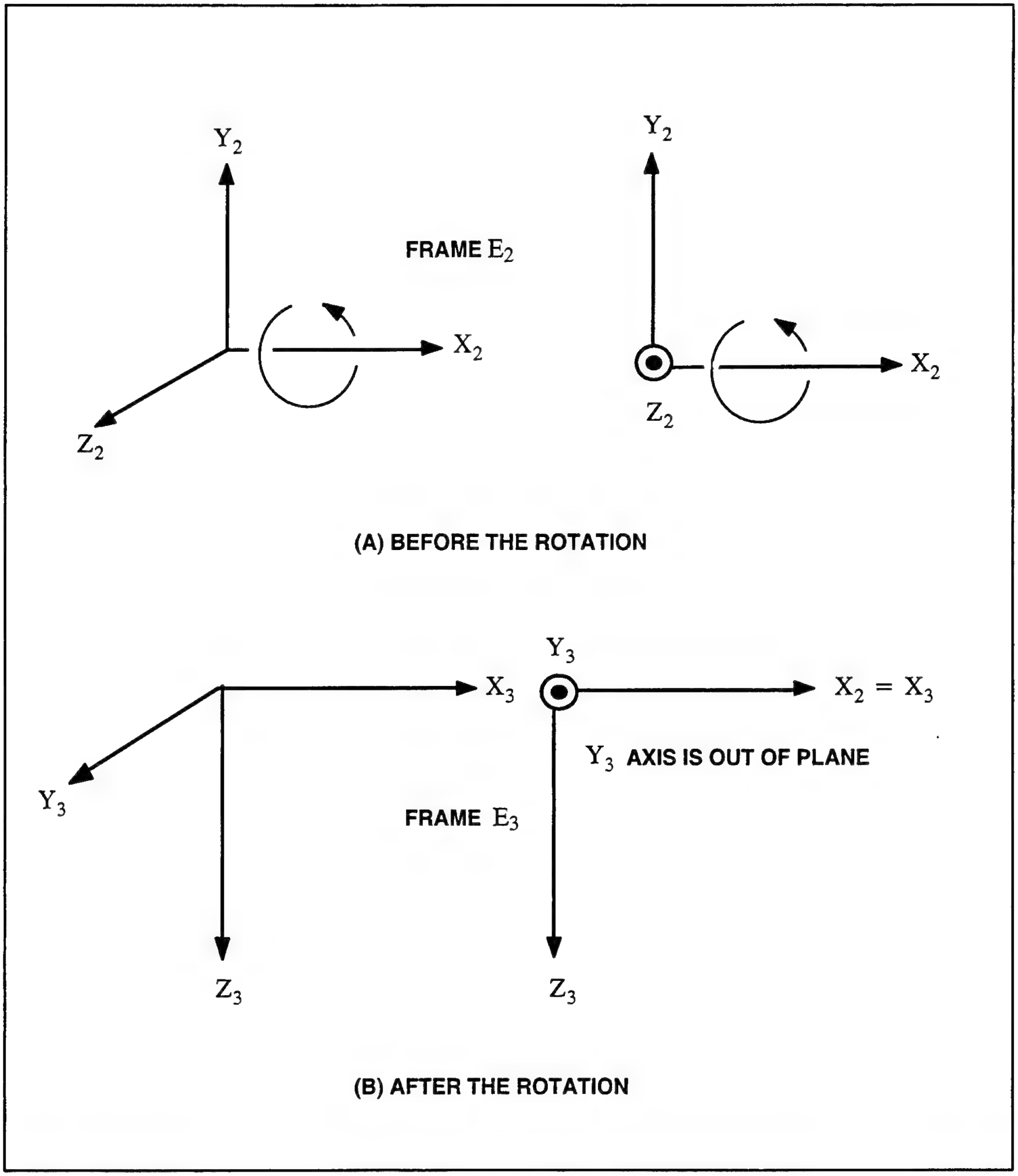


Figure 4-9. Third Rotation About the X_2–Axis in Helmholtz's System

4.3 ANGULAR ROTATIONS DO NOT COMMUTE

As briefly discussed in Section 1.2, adding two vectors, $\mathbf{V}_1$ and $\mathbf{V}_2$, we found out that $\mathbf{V}_1 + \mathbf{V}_2 = \mathbf{V}_2 + \mathbf{V}_1$, as shown in Figure 4-10:

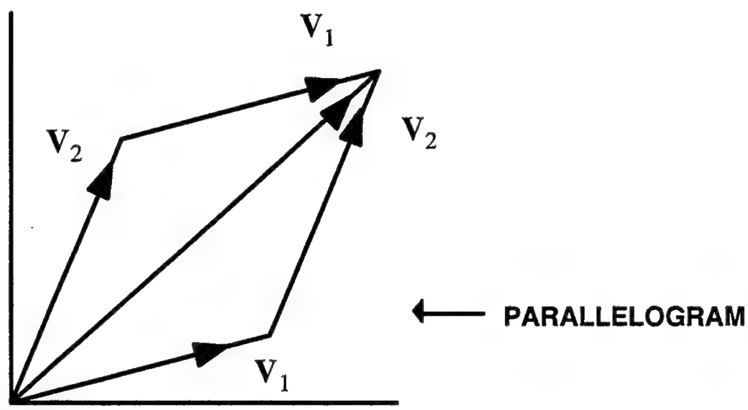
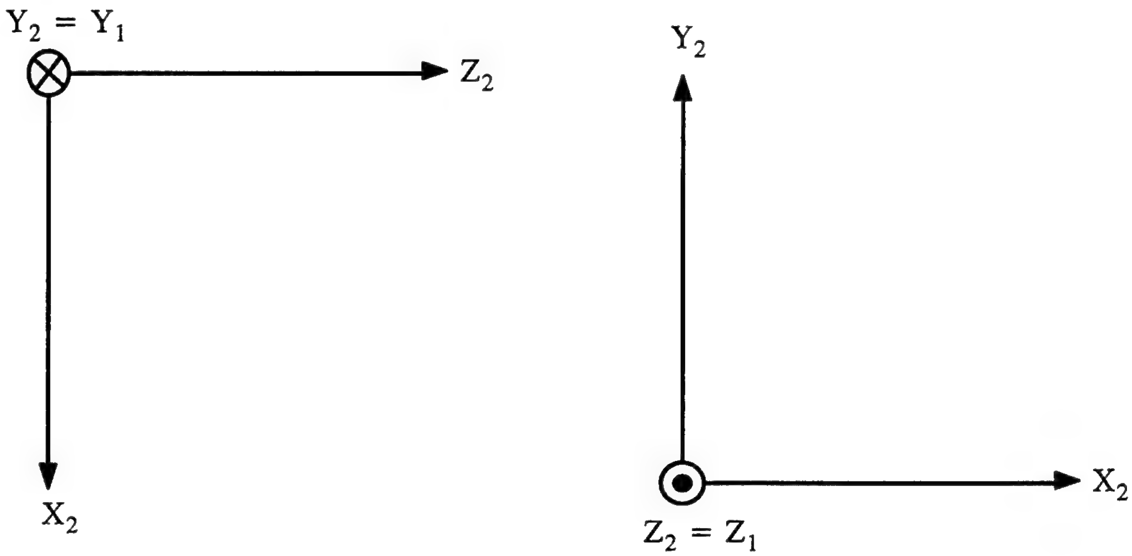


Figure 4-10. Addition of Two Vectors

That is, vector motion (addition) commutes. In contrast, the first two rotations of Fick's system (Z_0–axis rotation followed by Y_1–axis rotation) do not result in the same orientation produced by the first two rotations of Helmholtz's system (Y_0–axis rotation followed by Z_1–axis rotation) for the same amounts of the angular rotations, as repeated below in Figure 4-11, copied from Figures 4-5 and 4-8.



ORIENTATION OF EYE FRAME AFTER THE FIRST ROTATION ABOUT Z_0–AXIS ROTATION BY 90^0 FOLLOWED BY THE SECOND ROTATION ABOUT Y_1–AXIS BY 90^0 IN FICK'S SYSTEM (SEE FIGURE 4-5).

ORIENTATION OF EYE FRAME AFTER THE FIRST ROTATION ABOUT Y_0–AXIS BY 90^0 FOLLOWED BY THE SECOND ROTATION ABOUT Z_1–AXIS BY 90^0 IN HELMHOLTZ'S SYSTEM (SEE FIGURE 4-8).

Figure 4-11. Comparison of Eye Frame Orientation of Fick's System and Helmholtz's System After the Second Rotation

Mathematically, this means that for any Vectors $\mathbf{V}_1$ and $\mathbf{V}_2$, $\mathbf{V}_1 + \mathbf{V}_2 = \mathbf{V}_2 + \mathbf{V}_1$. But, for Rotation Matrices R_1 and R_2, generally $R_1R_2 \neq R_2R_1$. That is, the rotation corresponding to R_1 followed by the rotation corresponding to R_2 is not the same as R_2 followed by R_1. That is, R_1 and R_2 do not commute. This is discussed in Section 5.

Although finite rotations may not be considered as vectors and thus do not commute, infinitesimal rotations may be considered as such. That is, in the limit, as the angle of rotation becomes very small, Rotation Matrices commute as does vector addition. This is discussed in Section 7.

SECTION 5

THE ROTATION MATRIX

In the following pages, we will derive the Rotation Matrices for the three basic rotations around the Z–axis (yaw or horizontal rotation), the Y–axis (pitch or vertical rotation), and the X–axis (roll or torsional rotation). Any 3D rotation in space can be produced by a combination (multiplication) of these basic rotations.

5.1 BASIC ROTATION AROUND THE Z–AXIS

Consider a vector **r** in an orthogonal Frame A with components x_A, y_A, and z_A as shown in Figure 5-1.

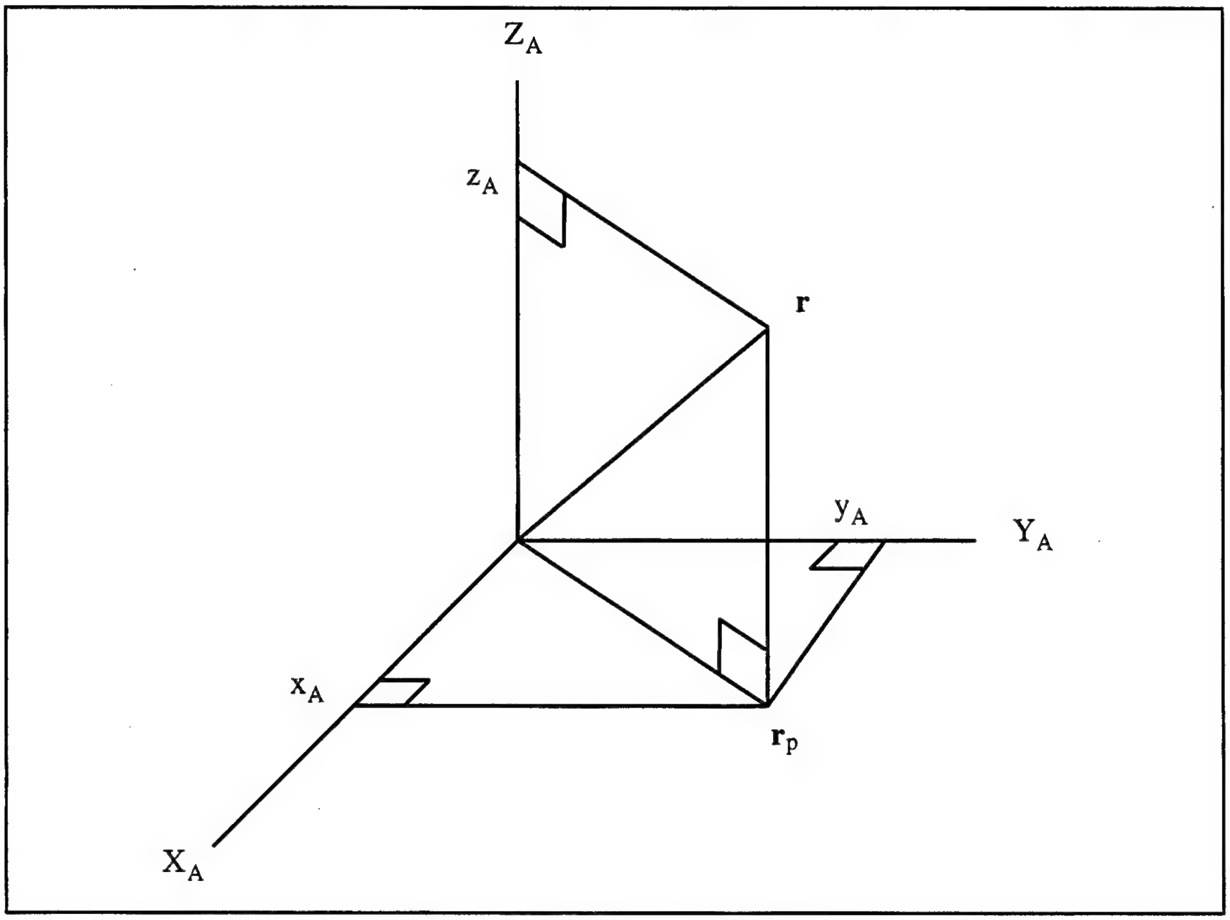


Figure 5-1. Frame A with Vector r

Frame B, which is initially coincident with Frame A, is rotated about the common $Z_A = Z_B$–axis counterclockwise by an angle θ_Z as shown in Figure 5-2. The same vector **r** has

components x_B, y_B, and z_B in Frame B. Our goal is to find a relationship between x_A, y_A, z_A and x_B, y_B, z_B of the same vector **r**.

Let $\mathbf{r}_p$ (in Figure 5-4) be the projection of the vector **r** in the Y–Z plane of both frames. Decomposing the components of $\mathbf{r}_p$ in X_A, Y_A, and X_B directions, we get Figures 5-3 and 5-4. We have a pair of identical right triangles with hypotenuse x_A, and another pair with hypotenuse y_A in Figure 5-4.

By inspection, we get, from Figures 5-3 and 5-4:

$$x_B = x_A \cos\theta_Z + y_A \sin\theta_Z \tag{5-1}$$

Since $y_A \cos\theta_Z = y_B + x_A \sin\theta_Z$, we get:

$$\begin{aligned} y_B &= y_A \cos\theta_Z - x_A \sin\theta_Z \\ &= -x_A \sin\theta_Z + y_A \cos\theta_Z \end{aligned} \tag{5-2}$$

Since the Z_A–axis coincides with the Z_B–axis,

$$z_B = z_A \tag{5-3}$$

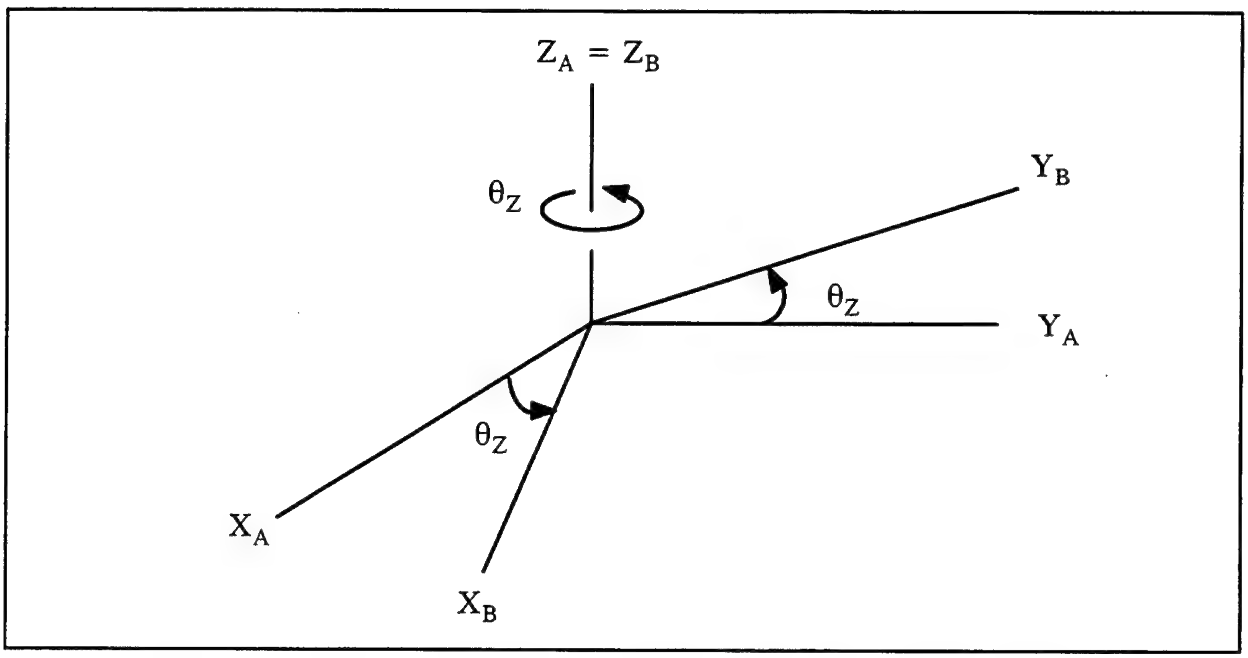


Figure 5-2. Frame A is Rotated to Frame B about the Z–axis by Angle θ_Z

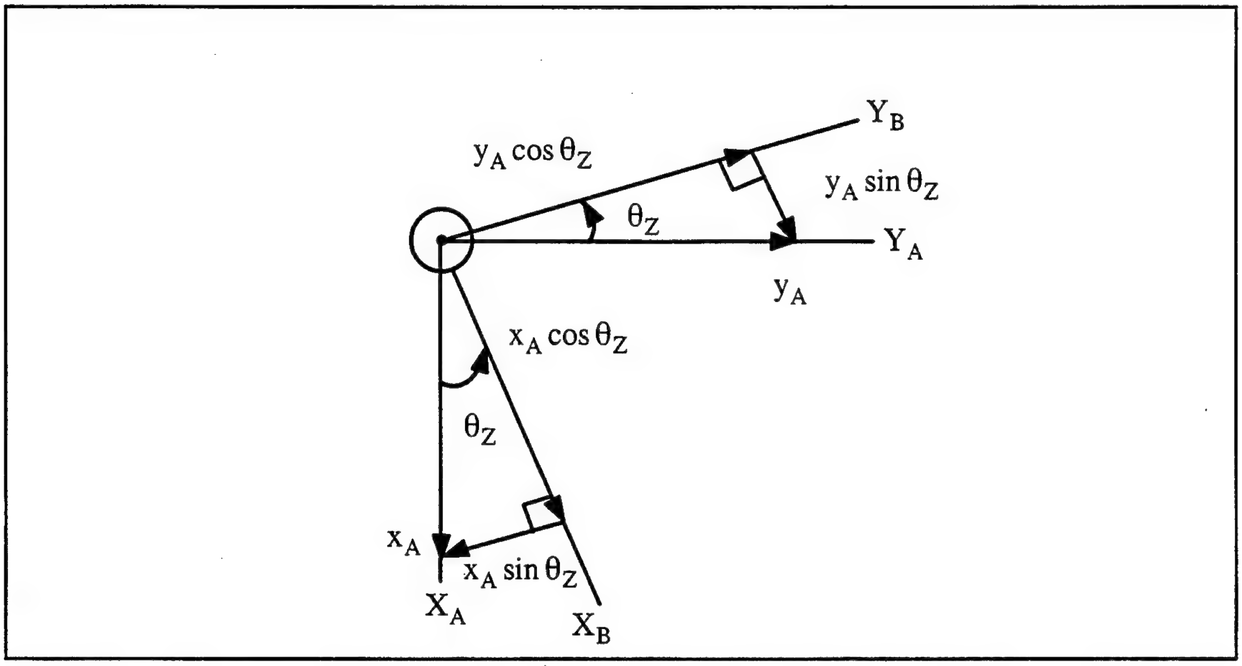


Figure 5-3. Components of r in Frame A decomposed into components in Frame B (see Figure 5-4)

$x_A = aj = dg \qquad y_A = ad = jg \qquad \mathbf{r}_P = ag$

$ai = x_A \cos\theta_Z \qquad fg = y_A \sin\theta_Z \qquad ed = x_A \sin\theta_Z \qquad ac = y_A \cos\theta_Z$

$x_B = ah = ai + ih = ai + fg = x_A \cos\theta_Z + y_A \sin\theta_z$ Equation (5-1)

$y_B = ab = ac - bc = ac - ed = y_A \cos\theta_Z - x_A \sin\theta_z$

$= -x_A \sin\theta_Z + y_A \cos\theta_z$ Equation (5-2)

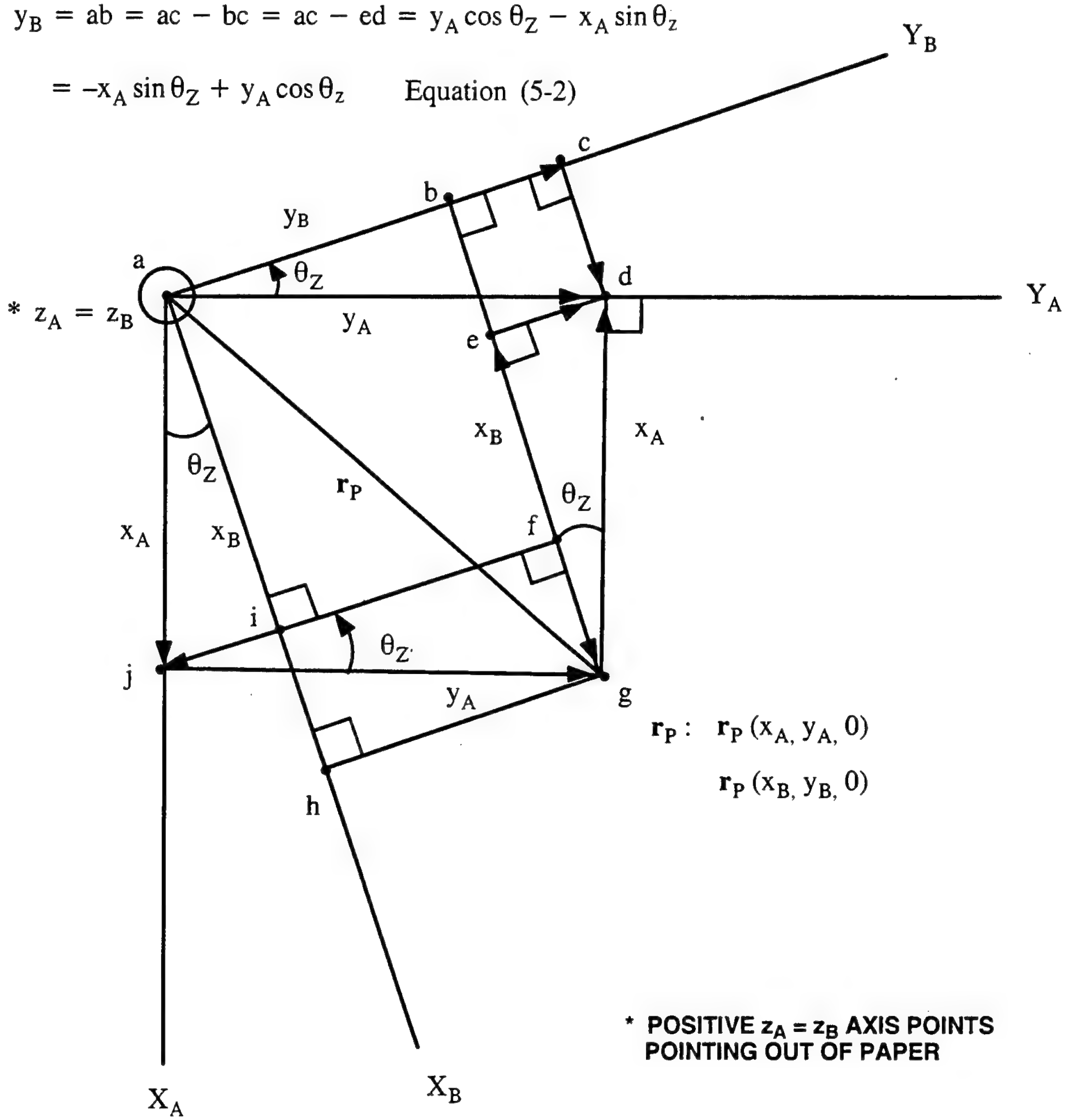


Figure 5-4. Components of r_p in both Frame A and Frame B

The coordinates x_A, y_A, z_A for **r** in Frame A may be represented either as a column vector $\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix}$ or as row vector $[x_A\ y_A\ z_B]$. Similarly, the coordinates x_B, y_B, z_B for **r** in Frame B may be represented either as $\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$ or as $[x_B\ y_B\ z_B]$.

If we use column vector representation for components, we get from Equations (5-1) to (5-3):

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \tag{5-4}$$

By denoting column vectors by:

$$\mathbf{r}^A = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} ;\ \text{and} \quad \mathbf{r}^B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \tag{5-5}$$

and denoting the Coefficient Matrix in Equation (5-4) by C_A^B, (from Frame A to Frame B), we have

$$C_A^B = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{5-6}$$

It follows:

$$\mathbf{r}^B = C_A^B\ \mathbf{r}^A \tag{5-7}$$

Superscript B in $\mathbf{r}^B$ and Superscript A in $\mathbf{r}^A$ indicate the frame in which we have resolved the vector.

The matrix C_A^B in Equation (5-7) transforms the coordinate of a vector expressed in one frame (Frame A) to another frame (Frame B). Therefore, it is called the "**Coordinate Transformation Matrix**." It is also called "**Rotation Matrix**" because it shows the effect of rotation of one frame (Frame B) relative to another (Frame A). It is also called "**Direction Cosine Matrix**" because the elements of the matrix are all cosines of angle θ $\left[\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)\right]$. The notation C_A^B indicates a transformation <u>from</u> A <u>to</u> B, or from the subscript to the superscript.

The determinant of C_A^B in Equation (5-7) denoted by $|C_A^B|$ is

$$|C_A^B| = \begin{vmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2\theta_z - (-\sin^2\theta_z)$$

$$= 1 \qquad (5\text{-}8)$$

It turns out that the determinant of any **Rotation Matrix** is always equal to 1.

Reversing the rotation process, the Frame B is rotated about the common $Z_A = Z_B$–axis clockwise by angle θ_z, bringing the Frame B back to position of coincidence with Frame A, as shown in Figure 5-5, thus reversing the process described in Figure 5-2.

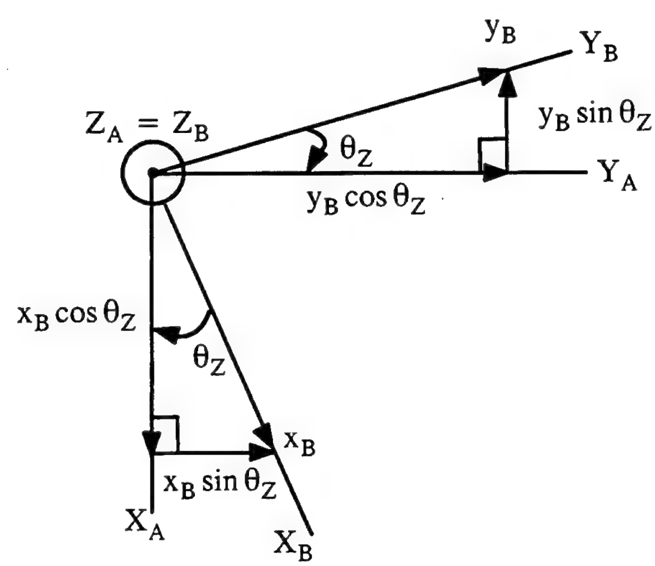


Figure 5-5. Frame B Rotated Clockwise Back to Frame A

Knowing θ_z, we want to determine x_A, y_A, z_A in Frame A of the vector **r** in terms of x_B, y_B, and z_B of **r** in Frame B.

Summing components in X_A, Y_A and Z_A directions:

$$x_A = x_B \cos\theta_Z - y_B \sin\theta_Z \tag{5-9}$$

$$y_A = x_B \sin\theta_Z + y_B \cos\theta_Z \tag{5-10}$$

$$z_A = z_B \tag{5-11}$$

Note that the $(- y_B \sin\theta_z)$ term in Equation (5-9) is negative because it is in the direction of negative X_A–axis direction.

If we use column vector representation for the components, we get from Equations (5-9), (5-10), and (5-11):

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \tag{5-12}$$

Using the notations given in Equation (5-5), and denoting the Coefficient Matrix in Equation (5-12) by C_B^A, (from Frame B to Frame A), we have:

$$C_B^A = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-13}$$

It follows that

$$\mathbf{r}^A = C_B^A \, \mathbf{r}^B \tag{5-14}$$

Substituting Equation (5-7) into Equation (5-14) for $\mathbf{r}^B$, we have

$$\mathbf{r}^A = C_B^A \, C_A^B \, \mathbf{r}^A \tag{5-15}$$

Since $\mathbf{r}^A$ is equal to itself, Equation (5-15) indicates that $C_B^A \, C_A^B$ must be the **Identity Matrix**:

$$C_B^A \, C_A^B = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-16}$$

This makes sense geometrically because Frame A is rotated to Frame B by C_A^B, and then Frame B is rotated back to Frame A by C_B^A. Therefore, Frame B is identical to or coincident with

Frame A. Note that the second rotation denoted by C_B^A is multiplied from the left. This occurs because the coordinates of the vectors are represented in column vector format as shown in Equation (5-5). We may confirm the validity of Equation (5-15) analytically by direct substitution:

Using Equation (5-13) and Equation (5-6):

$$C_B^A C_A^B = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta_z + \sin^2\theta_z & \cos\theta_z \sin\theta_z - \sin\theta_z \cos\theta_z & 0 \\ \sin\theta_z \cos\theta_z - \cos\theta_z \sin\theta_z & \sin^2\theta_z + \cos^2\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-17}$$

which validates the conclusion of Equation (5-16).

Referring C_A^B given Equation (5-6), if we exchange its rows and its columns, that is, if we take the transpose of C_A^B, then by denoting the transpose of C_A^B by $\left(C_A^B\right)^T$, we find $\left(C_A^B\right)^T = C_B^A$. That is,

$$\left[C_A^B\right]^T = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= C_B^A \tag{5-18}$$

which is the same as C_B^A given in Equation (5-13).

Substituting Equation (5-18) into Equation (5-16),

$$\left[C_A^B\right]^T C_A^B = I \tag{5-19}$$

Now substituting Equation (5-14) into Equation (5-7):

$$\mathbf{r}^B = C_A^B C_B^A \mathbf{r}^B \tag{5-20}$$

Since $\mathbf{r}^B$ is equal to itself, Equation (5-20) requires:

$$C_A^B C_B^A = I \tag{5-21}$$

Now, substituting Equation (5-18) into Equation (5-21) for C_B^A:

$$C_A^B \left[C_A^B\right]^T = I \tag{5-22}$$

By definition, the inverse of C denoted by, $C^{-1} = \left(C_A^B\right)^{-1}$ must satisfy $C^{-1}C = CC^{-1} = I$, or it must satisfy

$$\left[C_A^B\right]^{-1} C_A^B = C_A^B \left[C_A^B\right]^{-1} = I \tag{5-23}$$

From Equation (5-19) and Equation (5-22):

$$\left[C_A^B\right]^T C_A^B = C_A^B \left[C_A^B\right]^T = I \tag{5-24}$$

Comparing Equation (5-23) and Equation (5-24):

$$\left[C_A^B\right]^T = \left[C_A^B\right]^{-1} \tag{5-25}$$

Equation (5-25) states that the inverse of C_A^B is equal to the transpose of C_A^B.

For any Rotation Matrix C that transforms the coordinate of one orthogonal frame to another orthogonal frame, it is always true that

$$C^T = C^{-1} \tag{5-26}$$

A matrix that satisfies Equation (5-26) is called an **Orthogonal Matrix**.

This is very convenient because the determination of C^{-1}, which is generally complicated, may be computed simply by exchanging the rows and columns of C, or by flipping the matrix around the diagonal–axis for an Orthogonal Matrix.

From Equation (5-24) we have, using C for C_A^B:

$$C^T C = I = CC^T \tag{5-27}$$

Using the notations defined by the right side of Equation (5-6) for C_A^B, we have

$$CC^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = I \tag{5-28}$$

Equation (5-27) or Equation (5-28) is called the **orthogonality condition**. Although Equation (5-27) is derived from a special case, it is true for all Orthogonal Matrices. It is used to normalize and orthogonalize the C matrix, in case the elements C_{ij} determined by measurements or computations deviate from the condition specified by Equation (5-27) or Equation (5-28). In that case, this is geometrically equivalent to non–orthogonal frames. For example, Equation (5-28) is often used to orthogonalize the eye coil field and to correct for measurement errors. (See Sections 11 to 13.)

5.2 Z–AXIS ROTATION USING ROW VECTORS

If we represent the coordinate of the vector $(\mathbf{r}^A)^T$ by a row vector $[x_A\ y_A\ z_A]$ in Frame A and the coordinates of $(\mathbf{r}^B)^T$ by a row vector $[x_B\ y_B\ z_B]$ in Frame B, we get from Equations (5-1) to (5-3):

$$\begin{bmatrix} x_B & y_B & z_B \end{bmatrix} = \begin{bmatrix} x_A & y_A & z_A \end{bmatrix} \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-29}$$

In view of the definition given in Equation (5-5):

$$\begin{bmatrix} x_B & y_B & z_B \end{bmatrix} = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}^T = \left[\mathbf{r}^B\right]^T \tag{5-30}$$

$$\begin{bmatrix} x_A & y_A & z_A \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix}^T = \left[\mathbf{r}^A\right]^T \tag{5-31}$$

From Equations (5-29) to (5-31):

$$\left[\mathbf{r}^B\right]^T = \left[\mathbf{r}^A\right]^T R_A^B \tag{5-32}$$

where

$$R_A^B = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{5-33}$$

Comparing R_A^B in Equation (5-33) with C_A^B in Equation (5-6), we note that R_A^B is equal to the transpose of C_A^B:

$$R_A^B = \left[C_A^B\right]^T \tag{5-34}$$

$$C_A^B = \left[R_A^B\right]^T \tag{5-35}$$

From Equation (5-33):

$$|R_A^B| = \begin{vmatrix} \cos\theta_Z & -\sin\theta_Z & 0 \\ \sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2\theta_Z + \sin^2\theta_Z = 1 \tag{5-36}$$

It follows from Equation (5-8) and Equations (5-34) through (5-36),

$$|C| = |C^T| = |R| = |R^T| = 1 \tag{5-37}$$

Perhaps for historical or implementation–related reasons, the eye movement community seems to prefer the R matrix (described in Equations (5-29) through (5-33)) corresponding to row vector representation of coordinates, while virtually all other scientific and engineering communities prefer the C matrix (described in Equations (5-4) through (5-7)) corresponding to column vector representation of coordinates.

5.3 BASIC ROTATION AROUND THE Y–AXIS

Next, we consider the counterclockwise rotation around the common Y–axis of Frame B relative to Frame A by θ_y.

Following exactly the same procedure as before, we get

$$x_B = x_A \cos\theta_Y + y_A 0 - z_A \sin\theta_y$$

$$y_B = x_A o + y_A + z_A 0$$

$$z_B = x_A \sin\theta_Y + y_A 0 + z_A \cos\theta_Y \tag{5-38}$$

or

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \tag{5-39}$$

It follows, using notations given in Equation (5-5),

$$\mathbf{r}^B = C_A^B \mathbf{r}^A$$

in which C_A^B is given by:

$$C_A^B = \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \tag{5-40}$$

for Y–axis rotation by θ_Y.

If we use row vector representation, we get from Equation (5-38):

$$\begin{bmatrix} x_B & y_B & z_B \end{bmatrix} = \begin{bmatrix} x_A & y_A & z_A \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \tag{5-41}$$

or, in terms of notations defined in Equation (5-5):

$$\left[\mathbf{r}^B\right]^T = \left[\mathbf{r}^A\right]^T R_A^B \tag{5-42}$$

where

$$R_A^B = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \tag{5-43}$$

we note, comparing Equation (5-43) with Equation (5-40):

$$R_A^B = \left[C_A^B\right]^T = C_B^A \tag{5-44}$$

as before.

5.4 BASIC ROTATION AROUND THE X–AXIS

Finally, we consider the counterclockwise rotation around the (originally) common X–axis of Frame B relative to Frame A by θ_x.

Following exactly the same procedure as before, we get

$$\begin{aligned} x_B &= x_A + y_A 0 + z_A 0 \\ y_B &= x_A 0 + y_A \cos\theta_x + z_A \sin\theta_X \\ z_B &= x_A 0 - y_A \sin\theta_X + z_A \cos\theta_X \end{aligned} \tag{5-45}$$

From Equation (5-45),

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \tag{5-46}$$

$$\mathbf{r}^B = C_A^B \, \mathbf{r}^A \tag{5-47}$$

We may also express Equation (5-45), using row vector representation:

$$\begin{bmatrix} x_B & y_B & z_B \end{bmatrix} = \begin{bmatrix} x_A & y_A & z_A \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$$

$$\left[\mathbf{r}^B\right]^T = \left[\mathbf{r}^A\right]^T R_A^B \tag{5-48}$$

Since $[AB]^T$ is equal to $B^T A^T$ (see Appendix A), taking the transpose of Equation (5-47):

$$\left[\mathbf{r}^B\right]^T = \left[C_A^B \, \mathbf{r}^A\right]^T$$

$$\left[\mathbf{r}^B\right]^T = \left[\mathbf{r}^A\right]^T \left[C_A^B\right]^T \tag{5-49}$$

Comparing Equation (5-49) with Equation (5-48), we conclude

$$\left[C_A^B\right]^T = R_A^B \tag{5-50}$$

as shown in Equation (5-34).

Taking the transpose of Equation (5-50):

$$\left[\left[C_A^B\right]^T\right]^T = \left[R_A^B\right]^T$$

Since $\left[C_A^B\right]^T = C_B^A$ and $\left[C_B^A\right]^T = C_A^B$, we have:

$$C_A^B = \left[R_A^B\right]^T \tag{5-51}$$

as shown in Equation (5-35).

That is, C_A^B corresponding to the column vector representation is the transpose of R_A^B corresponding to the row vector representation.

5.5 COLUMN VECTOR REPRESENTATION OF TWO CONSECUTIVE ROTATIONS

Now suppose $\mathbf{r}^B$ in the Frame B is transformed to $\mathbf{r}^C$ in Frame C (not to be confused with Rotation Matrix C). Then, using the column vector representation:

$$\mathbf{r}^C = C_B^C\, \mathbf{r}^B \tag{5-52}$$

Substituting Equation (5-47) in Equation (5-52) for $\mathbf{r}^B$:

$$\begin{aligned}\mathbf{r}^C &= C_B^C\left[C_A^B\, \mathbf{r}^A\right] \\ &= C_B^C \quad C_A^B \quad \mathbf{r}^A\end{aligned} \tag{5-53}$$

It is also true that

$$\mathbf{r}^C = C_A^C \quad \mathbf{r}^A \tag{5-54}$$

From Equations (5-53) and (5-54), we get:

$$C_A^C = C_B^C\, C_A^B \tag{5-55}$$

Note that in Equation (5-55), the matrix C_B^C corresponding to the second rotation is multiplied **from the left**, and that, while the rotations are sequential and might appear additive, they are represented mathematically by multiplication of matrices. Also, note that sequence in which the rotations take place is important since, in general, $C_B^C\, C_A^B \neq C_A^B\, C_B^C$.

5.6 ROW VECTOR REPRESENTATION OF TWO CONSECUTIVE ROTATIONS

Next, using the row vector representation for the transformation from Frame B to Frame C:

$$\left[\mathbf{r}^{C}\right]^{T} = \left[\mathbf{r}^{B}\right]^{T} R_{B}^{C} \tag{5-56}$$

Substituting Equation (5-48) into Equation (5-56) for $(\mathbf{r}^{B})^{T}$:

$$\begin{aligned}\left[\mathbf{r}^{C}\right]^{T} &= \left[\left[\mathbf{r}^{A}\right]^{T} R_{A}^{B}\right] R_{B}^{C} \\ &= \left[\mathbf{r}^{A}\right]^{T} R_{A}^{B} R_{B}^{C}\end{aligned} \tag{5-57}$$

Since it is also true that

$$\left[\mathbf{r}^{C}\right]^{T} = \left[\mathbf{r}^{A}\right]^{T} R_{A}^{C} \tag{5-58}$$

we conclude from Equation (5-57) and Equation (5-58) that when Frame A is rotated to Frame B, and Frame B is rotated to Frame C, the rotations from Frame A to Frame C equate to:

$$R_{A}^{C} = R_{A}^{B} R_{B}^{C} \tag{5-59}$$

Note that in Equation (5-59), the matrix R_{B}^{C} for the second rotation based on the transformation of row vectors multiplies **from the right**, unlike the case of C_{B}^{C} which multiplies from the left as shown in Equation (5-55).

Using similar procedures, we can show that when Frame C is rotated to Frame D:

$$C_{A}^{D} = C_{C}^{D} C_{B}^{C} C_{A}^{B} \tag{5-60}$$

while

$$R_{A}^{D} = R_{A}^{B} R_{B}^{C} R_{C}^{D} \tag{5-61}$$

Taking the transpose of Equation (5-60), and using the formula $[AB]^T = B^T A^T$ (see Appendix A),

$$\left[C_A^D\right]^T = \left[C_C^D\, C_B^C\, C_A^B\right]^T$$

$$= \left[C_A^B\right]^T \left[C_C^D\, C_B^C\right]^T$$

$$= \left[C_A^B\right]^T \left[C_B^C\right]^T \left[C_C^D\right]^T \qquad (5\text{-}62)$$

Using Equation (5-34) and its extension in Equation (5-62):

$$R_A^D = R_A^B\, R_B^C\, R_C^D \qquad (5\text{-}63)$$

which is identical with Equation (5-61).

SECTION 6

ROTATION MATRICES FOR FICK'S SYSTEM AND HELMHOLTZ'S SYSTEM OF EYE ROTATIONS

In the following pages, we will derive the resultant Rotation Matrix corresponding to the three consecutive Euler angle rotations according to the Fick's system (discussed in Section 4.1). We will also derive the resultant Rotation Matrix corresponding to the three consecutive Euler angle rotations according to the Helmholtz's system (discussed in Section 4.2).

We assume the Head Frame and the Eye Frame are initially aligned with their X–axes pointed straight forward, their Y–axes horizontally left, and their Z–axes vertically upward.

The Rotation Matrices for the three basic rotations around the Z–axis (yaw or horizontal rotation), the Y–axis (pitch or vertical rotation), and the X–axis (roll or torsional rotation) was derived in Section 5.

6.1 ROTATION MATRIX FOR FICK'S SYSTEM

In Fick's system of Euler Angles, the first rotation of the Eye Frame is the Z_H–axis of the Head Frame; this results in the Frame F1 (F for Fick). The second rotation of the Eye Frame is about the Y_{F1}–axis of the Frame F1; this results in Frame F2. The third rotation of the Eye Frame is about the X_{F2}–axis of Frame F2; this results in the final orientation of the Eye Frame in Fick's system.

FICK ROTATIONS: (see Section 5)

1) First rotation about Z_H–axis by θ_Z:

Head Frame H becomes Frame F_1 (F for Fick)

$$\mathbf{r}^{F1} = C_H^{F1} \ \mathbf{r}^H \tag{6-1}$$

Using the X–axis rotation matrix:

$$\begin{bmatrix} x_{F1} \\ y_{F1} \\ z_{F1} \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-2}$$

2) Second rotation about Y_{F1}–axis by θ_Y:

Frame F_1 becomes Frame F_2

$$\mathbf{r}^{F2} = C_{F1}^{F2} \ \mathbf{r}^{F1} \tag{6-3}$$

Using the Y–axis Rotation Matrix:

$$\begin{bmatrix} x_{F2} \\ y_{F2} \\ z_{F2} \end{bmatrix} = \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_{F1} \\ y_{F1} \\ z_{F1} \end{bmatrix} \tag{6-4}$$

Substituting Equation (6-1) into Equation (6-3):

$$\mathbf{r}^{F2} = C_{F1}^{F2} \;\; C_{H}^{F1} \;\; \mathbf{r}^{H} \tag{6-5}$$

Notice that the matrix for the second rotation C_{F1}^{F2} is multiplying the matrix for the first rotation C_{H}^{F1} from the left.

Substituting from Equation (6-2) and Equation (6-4) into Equation (6-5):

$$\begin{bmatrix} x_{F2} \\ y_{F2} \\ z_{F2} \end{bmatrix} = \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-6}$$

3) Third rotation about X_{F2}–axis by θ_X:

Frame F_2 becomes Frame F_3 or the Eye's final Frame E.

$$\mathbf{r}^{E} = \mathbf{r}^{F3} = C_{F2}^{F3} \;\; \mathbf{r}^{F2} \tag{6-7}$$

Using the Z–axis rotation matrix:

$$\begin{bmatrix} x_{E} \\ y_{E} \\ z_{E} \end{bmatrix} = \begin{bmatrix} x_{F3} \\ y_{F3} \\ z_{F3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} x_{F2} \\ y_{F2} \\ z_{F2} \end{bmatrix} \tag{6-8}$$

Substituting Equation (6-5) into Equation (6-7) for $\mathbf{r}^{F2}$:

$$\mathbf{r}^{E} = C_{F2}^{F3} \;\; C_{F1}^{F2} \;\; C_{H}^{F1} \;\; \mathbf{r}^{H} \tag{6-9}$$

Notice again that C_{F2}^{F3} multiplies C_{F1}^{F2} from the left.

Substituting Equation (6-6) into Equation (6-8) for $\begin{bmatrix} x_{F2} \\ y_{F2} \\ z_{F2} \end{bmatrix}$:

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-10}$$

It follows:

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} \cos\theta_Y\cos\theta_Z & \cos\theta_Y\sin\theta_Z & -\sin\theta_Y \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ \sin\theta_Y\cos\theta_Z & \sin\theta_Y\sin\theta_Z & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-11}$$

which becomes:

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} \cos\theta_Y\cos\theta_Z & \cos\theta_Y\sin\theta_Z & -\sin\theta_Y \\ -\cos\theta_X\sin\theta_Z + \sin\theta_X\sin\theta_Y\cos\theta_Z & \cos\theta_X\cos\theta_Z + \sin\theta_X\sin\theta_Y\sin\theta_Z & \sin\theta_X\cos\theta_Y \\ \sin\theta_X\sin\theta_Z + \cos\theta_X\sin\theta_Y\cos\theta_Z & -\sin\theta_X\cos\theta_Z + \cos\theta_X\sin\theta_Y\sin\theta_Z & \cos\theta_X\cos\theta_Y \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-12}$$

Equation (6-9) may also be expressed as:

$$\mathbf{r}^E = C_H^{F3}\,\mathbf{r}^H = C_H^E\,\mathbf{r}^H \tag{6-13}$$

where C_H^E is given by the Coefficient Matrix on the right side of Equation (6-12).

Taking the transpose of Equation (6-13)

$$\left[\mathbf{r}^{E}\right]^{T} = \left[C_{H}^{E}\,\mathbf{r}^{H}\right]^{T}$$

$$= \left[\mathbf{r}^{H}\right]^{T}\left[C_{H}^{E}\right]^{T}$$

or

$$\left[\mathbf{r}^{E}\right]^{T} = \left[\mathbf{r}^{H}\right]^{T} R_{H}^{E} \tag{6-14}$$

Equation (6-14) may be written as

$$\left[x_E \quad y_E \quad z_E\right] = \left[x_H \quad y_H \quad z_H\right] \; R_{H}^{E} \tag{6-15}$$

where $R_H^E = \left(C_H^E\right)^T$ is the transpose of the Coefficient Matrix C_H^E of Equation (6-12) for the Fick's System, which is given below in Equation (6-16).

$$R_{H}^{E} = \left[C_{H}^{E}\right]^{T} \tag{6-16}$$

$$\begin{bmatrix} \cos\theta_Y\cos\theta_Z & -\cos\theta_X\sin\theta_Z + \sin\theta_X\sin\theta_Y\cos\theta_Z & \sin\theta_X\sin\theta_Z + \cos\theta_X\sin\theta_Y\cos\theta_Z \\ \cos\theta_Y\sin\theta_Z & \cos\theta_X\cos\theta_Z + \sin\theta_X\sin\theta_Y\sin\theta_Z & -\sin\theta_X\cos\theta_Z + \cos\theta_X\sin\theta_Y\sin\theta_Z \\ -\sin\theta_Y & \sin\theta_X\cos\theta_Y & \cos\theta_X\cos\theta_Y \end{bmatrix}$$

for the Fick's system of Euler Angle Rotations. (This is the transpose (rows and columns exchange) of the Coefficient Matrix of Equation (6-12).)

Now, taking the transpose of Equation (6-9) (see Appendix A):

$$\left[\mathbf{r}^{E}\right]^{T} = \left[C_{F2}^{F3}\,C_{F1}^{F2}\,C_{H}^{F1}\,\mathbf{r}^{H}\right]^{T}$$

$$= \left[\mathbf{r}^{H}\right]^{T}\left[\left[C_{F2}^{F3}\,C_{F1}^{F2}\right]C_{H}^{F1}\right]^{T}$$

$$= \left[\mathbf{r}^{H}\right]^{T}\left[C_{H}^{F1}\right]^{T}\left[C_{F2}^{F3}\,C_{F1}^{F2}\right]^{T}$$

$$= \left[\mathbf{r}^{H}\right]^{T}\left[C_{H}^{F1}\right]^{T}\left[C_{F1}^{F2}\right]^{T}\left[C_{F2}^{F3}\right]^{T} \tag{6-17}$$

or

$$\left[\mathbf{r}^{E}\right]^{T} = \left[\mathbf{r}^{H}\right]^{T}\left[R_{H}^{F1}\, R_{F1}^{F2}\, R_{F2}^{F3}\right] \tag{6-18}$$

or

$$\begin{aligned}\left[\mathbf{r}^{E}\right]^{T} &= \left[\mathbf{r}^{H}\right]^{T} R_{H}^{F3} \\ &= \left[\mathbf{r}^{H}\right]^{T} R_{H}^{E}\end{aligned} \tag{6-19}$$

which is equal to Equation (6-14) and Equation (6-15).

6.2 ROTATION MATRIX FOR HELMHOLTZ'S SYSTEM

In Helmholtz's system of Euler Angles, the first rotation of the Eye Frame is around the Y_H–axis of the Head Frame. This results in Frame H_1 (H for Helmholtz). The second rotation of the Eye Frame is around the Z_{H1}–axis of the Frame H_1. This results in the Frame H_2. The third rotation of the Eye Frame is around X_{H2}–axis of the Frame H_2. This results in the final orientation of the Eye Frame in Helmholtz's system.

HELMHOLTZ ROTATIONS: (see Section 5)

1) First rotation around the Y_H–axis by θ_Y:

Frame H becomes Frame H_1 :

$$\mathbf{r}^{H1} = C_{H}^{H1} \quad \mathbf{r}^{H} \tag{6-20}$$

Using the Y–axis rotation matrix:

$$\begin{bmatrix} x_{H1} \\ y_{H1} \\ z_{H1} \end{bmatrix} = \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-21}$$

2) Second rotation around Z_{H1}–axis by θ_Z:

Frame H_1 becomes Frame H_2

$$\mathbf{r}^{H2} = C_{H1}^{H2} \quad \mathbf{r}^{H1} \tag{6-22}$$

Using the Z–axis rotation matrix:

$$\begin{bmatrix} x_{H2} \\ y_{H2} \\ z_{H2} \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{H1} \\ y_{H1} \\ z_{H1} \end{bmatrix} \tag{6-23}$$

Substituting Equation (6-20) into Equation (6-22):

$$\mathbf{r}^{H2} = C_{H1}^{H2} \; C_{H}^{H1} \; \mathbf{r}^{H} \tag{6-24}$$

Substituting Equation (6-21) and Equation (6-23) into Equation (6-24):

$$\begin{bmatrix} x_{H2} \\ y_{H2} \\ z_{H2} \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_{H} \\ y_{H} \\ z_{H} \end{bmatrix} \tag{6-25}$$

3) Third rotation around X_{H2}–axis by θ_X:

Frame H_2 becomes Frame H_3 or the Eye's final Frame E.

$$\mathbf{r}^{E} = \mathbf{r}^{H3} = C_{H2}^{H3} \; \mathbf{r}^{H2} \tag{6-26}$$

Using the X–axis rotation matrix:

$$\begin{bmatrix} x_{E} \\ y_{E} \\ z_{E} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} x_{H2} \\ y_{H2} \\ z_{H2} \end{bmatrix} \tag{6-27}$$

Substituting Equation (6-24) into Equation (6-26):

$$\mathbf{r}^{E} = C_{H2}^{H3} \; C_{H1}^{H2} \; C_{H}^{H1} \; \mathbf{r}^{H} \tag{6-28}$$

Substituting from Equation (6-21), Equation (6-23) and Equation (6-27) into Equation (6-28):

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-29}$$

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} \cos\theta_Z\cos\theta_Y & \sin\theta_Z & -\cos\theta_Z\sin\theta_Y \\ -\sin\theta_Z\cos\theta_Y & \cos\theta_Z & \sin\theta_Z\sin\theta_Y \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-30}$$

$$\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} = \begin{bmatrix} \cos\theta_Z\cos\theta_Y & \sin\theta_Z & -\cos\theta_Z\sin\theta_Y \\ -\cos\theta_X\sin\theta_Z\cos\theta_Y + \sin\theta_X\sin\theta_Y & \cos\theta_X\cos\theta_Z & \cos\theta_X\sin\theta_Z\sin\theta_Y + \sin\theta_X\cos\theta_Y \\ \sin\theta_X\sin\theta_Z\cos\theta_Y + \cos\theta_X\sin\theta_Y & -\sin\theta_X\cos\theta_Z & -\sin\theta_X\sin\theta_Z\sin\theta_Y + \cos\theta_X\cos\theta_Y \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} \tag{6-31}$$

$$\mathbf{r}^E = C_H^{H3}\,\mathbf{r}^H = C_H^E\,\mathbf{r}^H \tag{6-32}$$

for the Helmholtz's system, where C_H^E is given by Coefficient Matrix of the right side of Equation (6-31).

Taking the transpose of Equation (6-32):

$$\left[\mathbf{r}^E\right]^T = \left[C_H^E\,\mathbf{r}^H\right]^T$$

$$= \left[\mathbf{r}^H\right]^T \left[C_H^E\right]^T \tag{6-33}$$

It follows

$$\left[\mathbf{r}^E\right]^T = \left[\mathbf{r}^H\right]^T R_H^E \tag{6-34}$$

since $\left[C_H^E\right]^T = R_H^E$.

Equation (6-34) may be written as

$$\left[x_E\; y_E\; z_E\right] = \left[x_H\; y_H\; z_H\right] R_H^E \tag{6-35}$$

where $R_H^E = \left(C_H^E\right)^T$ is the transpose of the Coefficient Matrix C_H^E of Equation (6-31) for the Helmholtz's system, as given in Equation (6-36):

$$R_H^E = \left[C_H^E\right]^T = \begin{bmatrix} \cos\theta_Z \cos\theta_Y & -\cos\theta_X \sin\theta_Z \cos\theta_Y + \sin\theta_X \sin\theta_Y & \sin\theta_X \sin\theta_Z \cos\theta_Y + \cos\theta_X \sin\theta_Y \\ \sin\theta_Z & \cos\theta_X \cos\theta_Z & -\sin\theta_X \cos\theta_Z \\ -\cos\theta_Z \sin\theta_Y & \cos\theta_X \sin\theta_Z \sin\theta_Y + \sin\theta_X \cos\theta_Y & -\sin\theta_X \sin\theta_Z \sin\theta_Y + \cos\theta_X \cos\theta_Y \end{bmatrix} \tag{6-36}$$

SECTION 7

ROTATION MATRIX FOR SMALL ANGLES

Although the Rotation Matrix does not commute for finite angles, it does commute for very small angles for which the approximations $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$, and $\theta\theta \simeq 0$ are justified. Consider the Rotation Matrix C_H^E (not R_H^E, which is $(C_H^E)^T$) of Fick's system. Using the above approximations, we have for small angle rotations of Fick's system,

$$C_H^E = \begin{bmatrix} 1 & \theta_Z & -\theta_Y \\ -\theta_Z + \theta_X\theta_Y & 1 + \theta_X\theta_Y\theta_Z & \theta_X \\ \theta_X\theta_Z + \theta_Y & -\theta_X + \theta_Y\theta_Z & 1 \end{bmatrix} \tag{7-1}$$

Since for very small angles $\theta_X\theta_Y \simeq 0$, $\theta_X\theta_Y\theta_Z \simeq 0$, $\theta_X\theta_Z \simeq 0$, and $\theta_Y\theta_Z \simeq 0$, it follows from Equation (7-1):

$$C_H^E = \begin{bmatrix} 1 & \theta_Z & -\theta_Y \\ -\theta_Z & 1 & \theta_X \\ \theta_Y & -\theta_X & 1 \end{bmatrix} \tag{7-2}$$

for Fick's system.

Now, consider the Rotation Matrix C_H^E (not R_H^E, which is $(C_H^E)^T$) of Helmholtz's system using the approximations $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$, and $\theta\theta \simeq 0$ for small angle rotations:

$$C_H^E = \begin{bmatrix} 1 & \theta_Z & -\theta_Y \\ -\theta_Z + \theta_X\theta_Y & 1 & \theta_Z\theta_Y + \theta_X \\ \theta_X\theta_Z + \theta_Y & -\theta_X & -\theta_X\theta_Z\theta_Y + 1 \end{bmatrix} \tag{7-3}$$

for Helmholtz's system. Note that Equation (7-3) is different from Equation (7-1). However, with approximations $\theta_x\theta_y \simeq \theta_x\theta_z \simeq \theta_z\theta_y \simeq \theta_x\theta_y\theta_z \simeq 0$, it may be reduced to:

$$C_H^E = \begin{bmatrix} 1 & \theta_Z & -\theta_Y \\ -\theta_Z & 1 & \theta_X \\ \theta_Y & -\theta_X & 1 \end{bmatrix} \tag{7-4}$$

which is identical to Equation (7-2). Equation (7-4) is the Rotation Matrix for the Helmholtz's system with angles small enough to justify $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$.

Thus, for small angles,

$$C_H^E \text{ (for Fick's System)} = C_H^E \text{ (for Helmholtz's System)}$$

$$= \begin{bmatrix} 1 & \theta_Z & -\theta_Y \\ -\theta_Z & 1 & \theta_X \\ \theta_Y & -\theta_X & 1 \end{bmatrix} \quad (7\text{-}5)$$

That is, for small angles, the sequence of rotation does not matter, and; therefore, the Rotation Matrices commute, or

$$C_{F2}^{F3}(\theta_X)\,C_{F1}^{F2}(\theta_Y)\,C_H^{F1}(\theta_Z) = C_{H2}^{H3}(\theta_X)\,C_{H1}^{H2}(\theta_Z)\,C_H^{H1}(\theta_Y)$$

making the final eye positions equivalent in both Helmholtz's and Fick's systems for the same amount of angular rotations. That is, for the separate rotations C_1, C_2 and C_3:

$$C_1C_2C_3 = C_1C_3C_2 = C_2C_1C_3 = C_2C_3C_1 = C_3C_1C_2 = C_3C_2C_1 \quad (7\text{-}6)$$

for small angle rotations.

SECTION 8

A DIFFERENT VIEW OF ROTATION MATRIX

Consider a vector **r** has components x_A, y_A and z_A in Frame A, and components x_B, y_B, and z_B in Frame B. Frame B is displaced **counterclockwise** from Frame A (reference frame) by a rotation angle θ_Z about the common Z–axis. For this case, we found out the Rotation Matrix relating $\mathbf{r}^A = [x_A\ y_A\ z_A]^T$ to $\mathbf{r}^B = [x_B\ y_B\ z_B]^T$ is given by (refer to Section 5):

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \tag{8-1}$$

Now instead of rotating Frame B relative to Frame A, a vector $\mathbf{r}_1$ in Frame A with coordinates x_1, y_1, and z_1 is rotated **clockwise** about the negative z–axis, according to the right hand rule to a new position $\mathbf{r}_2$ in the same frame with coordinates x_2, y_2, and z_2 by the angle θ_Z. This means $\mathbf{r}_1$ is moved downward on the X–Y plane as shown in Figure 8-1.

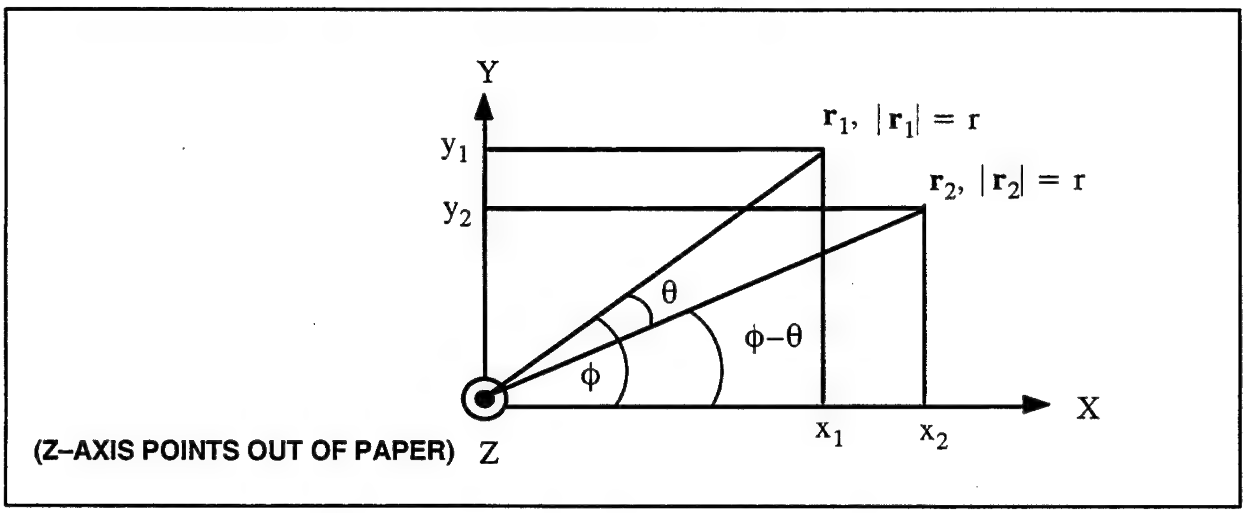


Figure 8-1. Clockwise Rotation of Vector $\mathbf{r}_1$

Referring to Figure 8-1, θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$. ϕ is the angle between $\mathbf{r}_1$ and the X–axis. Note that $\mathbf{r}_1$ and $\mathbf{r}_2$ have the same magnitude, which we denote by r.

It follows:

$$x_1 = r\cos\phi \tag{8-2}$$

$$y_1 = r\sin\phi \tag{8-3}$$

Using the trigonometry formula for $\cos(\phi-\theta)$ and $\sin(\phi-\theta)$:

$$
\begin{aligned}
x_2 &= r\cos(\phi-\theta) \\
&= r(\cos\phi\cos\theta + \sin\phi\sin\theta) \\
&= \cos\theta\,(r\cos\phi) + \sin\theta\,(r\sin\phi) \\
&= (\cos\theta)x_1 + (\sin\theta)y_1
\end{aligned} \tag{8-4}
$$

Using Equation (8-2) and Equation (8-3):

$$
\begin{aligned}
y_2 &= r\sin(\phi-\theta) \\
&= r(\sin\phi\cos\theta - \cos\phi\sin\theta) \\
&= \cos\theta\,(r\sin\phi) - \sin\theta\,(r\cos\phi) \\
&= (\cos\theta)y_1 - (\sin\theta)x_1 \\
&= -(\sin\theta)x_1 + (\cos\theta)y_1
\end{aligned} \tag{8-5}
$$

$$z_1 = z_2 \tag{8-6}$$

It follows:

$$
\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{8-7}
$$

Comparing the Coefficient Matrix of Equation (8-7) with that of Equation (8-1), we notice that they are identical.

Generalizing the above results, the Rotation Matrix corresponding to the rotation of one frame relative to the reference frame is the same as the Rotation Matrix corresponding to the rotation of a vector in the opposite direction (with same angle of rotation) within the original reference frame. Most of the time, the Rotation Matrix is used in the former context. However, if it is used in the latter context, it should be made explicit to avoid confusion.

SECTION 9

DETERMINATION OF ROTATION ANGLES FROM ROTATION MATRIX OF FICK'S SYSTEM

Until now we have defined angular rotations and, from these rotations, have determined the elements of a Rotation Matrix. Now we reverse the process, and assume we have a Rotation Matrix and wish to obtain the rotation angles that correspond to the matrix.

9.1 ANGLES OF FICK'S SYSTEM MATRIX (SOME OF THIS SECTION HAS BEEN DISCUSSED IN AN EARLIER SECTION)

Assume the Eye Frame is rotated to another orientation by rotating the initial orientation around its X–axis by angle T (for Torsion). The corresponding Rotation Matrix $C_X(T)$ is given by:

$$C_X(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos T & \sin T \\ 0 & -\sin T & \cos T \end{bmatrix} \tag{9-1}$$

which is Equation (5-46).

The Rotation Matrix $C_Y(V)$ around the Y–axis by angle V (for vertical motion of the Eye–axis) is given by:

$$C_Y(V) = \begin{bmatrix} \cos V & 0 & -\sin V \\ 0 & 1 & 0 \\ \sin V & 0 & \cos V \end{bmatrix} \tag{9-2}$$

which is Equation (5-39).

The Rotation Matrix $C_Z(H)$ around the Z–axis by angle H (for horizontal motion of the Eye–axis) is given by:

$$C_Z(H) = \begin{bmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9-3}$$

which is Equation (5-4).

By definition, in Fick's system, the sequence of rotations is Z–axis rotation followed by Y–axis rotation, followed by X–axis rotation. Thus, using column–vector representation, the final Rotation Matrix C in Fick's system is given by:

$$C = C_X(T)\, C_Y(V)\, C_Z(H) = C(T, V, H) \tag{9-4}$$

Using Equation (9-2) and Equation (9-3),

$$C_Y(V)\, C_Z(H) = \begin{bmatrix} \cos V & 0 & -\sin V \\ 0 & 1 & 0 \\ \sin V & 0 & \cos V \end{bmatrix} \begin{bmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos V \cos H & \cos V \sin H & -\sin V \\ -\sin H & \cos H & 0 \\ \sin V \cos H & \sin V \sin H & \cos V \end{bmatrix} \tag{9-5}$$

Using Equation (9-1) and Equation (9-5),

$$C = C_X(T)\left(C_Y(V)\, C_Z(H)\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos T & \sin T \\ 0 & -\sin T & \cos T \end{bmatrix} \begin{bmatrix} \cos V \cos H & \cos V \sin H & -\sin V \\ -\sin H & \cos H & 0 \\ \sin V \cos H & \sin V \sin H & \cos V \end{bmatrix} \tag{9-6}$$

$$= \begin{bmatrix} \cos V \cos H & \cos V \sin H & -\sin V \\ -\cos T \sin H + \sin T \sin V \cos H & \cos T \cos H + \sin T \sin V \sin H & \sin T \cos V \\ \sin T \sin H + \cos T \sin V \cos H & -\sin T \cos H + \cos T \sin T \sin H & \cos T \cos V \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{9-7}$$

The Equation (9-7) is the final Rotation Matrix of Fick's system. Equation (9-7) is the same as the Coefficient Matrix derived in Equation (6-12). From Equation (9-7):

$$C_{13} = -\sin V = \sin(-V)$$

or

$$\sin^{-1} C_{13} = -V$$

or

$$V = -\sin^{-1} C_{13} \tag{9-8}$$

Thus, we can determine the value of V from the value of C_{13} determined from experiments. Next, from Equation (9-7):

$$C_{12} = \cos V \sin H$$

$$\sin H = \frac{C_{12}}{\cos V}$$

or

$$H = \sin^{-1}\left[\frac{C_{12}}{\cos V}\right] \tag{9-9}$$

where cos V is determined using V given in Equation (9-8). Note that, both in Fick's system and Helmholtz's system, all angles of eye rotations are physiologically constrained to be much less than 90°. Therefore, cos V will never be 0.

Now, from Equation (9-7):

$$C_{33} = \cos T \cos V$$

or

$$\cos T = \frac{C_{33}}{\cos V}$$

or

$$T = \cos^{-1}\left[\frac{C_{33}}{\cos V}\right] \tag{9-10}$$

The Torsion angle T may also be determined from Equation (9-7):

$$C_{23} = \sin T \cos V$$

$$\sin T = \frac{C_{23}}{\cos V}$$

$$T = \sin^{-1} \frac{C_{23}}{\cos V} \tag{9-11}$$

Now we know $C = R^T$.

That is:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{9-12}$$

From (9-12), we get:

$$C_{13} = R_{31;}\ C_{12} = R_{21;}\ C_{23} = R_{32;}\ C_{33} = R_{33} \tag{9-13}$$

From Equation (9-9) and Equation (9-13),

$$H = \sin^{-1} \left(\frac{R_{21}}{\cos V}\right) \tag{9-14}$$

Similarly,

$$T = \cos^{-1}\left(\frac{R_{33}}{\cos V}\right) \quad \text{or} \quad T = \sin^{-1}\left(\frac{R_{32}}{\cos V}\right) \tag{9-15}$$

$$V = -\sin^{-1} R_{31} \tag{9-16}$$

SECTION 10

DETERMINATION OF ROTATION ANGLES FROM ROTATION MATRIX OF HELMHOLTZ'S SYSTEM

10.1 ANGLES OF HELMHOLTZ'S SYSTEM MATRIX

By definition, in the Helmholtz's System, the sequence of rotation is: Y–axis rotation, followed by Z–axis rotation, followed by X–axis rotation. Thus, the final Rotation Matrix C in the Helmholtz's System is given by, using the notations described in the previous section:

$$C = C_X(T)C_Z(H)C_Y(V) \qquad (10\text{-}1)$$

Now:

$$C_Z(H)C_Y(V) = \begin{bmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos V & 0 & -\sin V \\ 0 & 1 & 0 \\ \sin V & 0 & \cos V \end{bmatrix} \qquad (10\text{-}2)$$

$$= \begin{bmatrix} \cos H \cos V & \sin H & -\cos H \sin V \\ -\sin H \cos V & \cos H & \sin H \sin V \\ \sin V & 0 & \cos V \end{bmatrix} \qquad (10\text{-}3)$$

It follows:

$$C_X(T)C_Z(H)C_Y(V) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos T & \sin T \\ 0 & -\sin T & \cos T \end{bmatrix} \begin{bmatrix} \cos H \cos V & \sin H & -\cos H \sin V \\ -\sin H \cos V & \cos H & \sin H \sin V \\ \sin V & 0 & \cos V \end{bmatrix} \tag{10-4}$$

$$= \begin{bmatrix} \cos H \cos V & \sin H & -\cos H \sin V \\ -\cos T \sin H \cos V + \sin T \sin V & \cos T \cos H & \cos T \sin H \sin V + \sin T \cos V \\ \sin T \sin H \cos V + \cos T \sin V & -\sin T \cos H & -\sin T \sin H \sin V + \cos T \cos V \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{10-5}$$

for Helmholtz's System. Equation (10-5) is the same coefficient matrix as that derived in Equation (6-31).

From Equation (10-5):

$$C_{12} = \sin H$$

$$H = \sin^{-1} C_{12} \tag{10-6}$$

$$C_{13} = -\cos H \sin V = \cos H \sin(-V)$$

$$\sin(-V) = \frac{C_{13}}{\cos H}$$

$$(-V) = \sin^{-1} \frac{C_{13}}{\cos H}$$

$$V = -\sin^{-1} \frac{C_{13}}{\cos H} \tag{10-7}$$

where cos H is determined using H in Equation (10-6). Note that, both in Fick's System and Helmholtz's System, all angles of eye rotations are constrained to be less than 90°, so that cos H will never be 0.

Next,

$$C_{22} = \cos T \cos H$$

$$\cos T = \frac{C_{22}}{\cos H}$$

$$T = \cos^{-1} \frac{C_{22}}{\cos H} \tag{10-8}$$

Torsion T may also be found from:

$$C_{32} = -\sin T \cos H$$

$$C_{32} = \sin(-T) \cos H$$

$$\sin(-T) = \frac{C_{32}}{\cos H}$$

$$(-T) = \sin^{-1} \frac{C_{32}}{\cos H}$$

$$T = -\sin^{-1} \frac{C_{32}}{\cos H} \tag{10-9}$$

Now, we know:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \tag{10-10}$$

From (10-10):

$$C_{12} = R_{21};\ C_{13} = R_{31};\ C_{22} = R_{22};\ C_{32} = R_{23} \tag{10-11}$$

It follows from (10-6) to (10-9), using (10-11):

$$H = \sin^{-1} R_{21} \tag{10-12}$$

$$V = -\sin^{-1} \frac{R_{31}}{\cos H} \tag{10-13}$$

$$T = \cos^{-1} \frac{R_{22}}{\cos H} \quad \text{or} \quad T = -\sin^{-1} \frac{R_{23}}{\cos H}\,. \tag{10-14}$$

SECTION 11

ORTHOGONAL MATRIX AND ORTHOGONALITY CONDITION

In Section 5, we have shown by means of a simple example based on one axis rotation that the **Coordinate Transformation Matrix (Rotation Matrix)** from Frame A to Frame B, denoted by C_A^B, has a property that its inverse is equal to its transpose. That is,

$$\left(C_A^B\right)^{-1} = \left(C_A^B\right)^T. \tag{11-1}$$

where C_A^B is an **Orthogonal Matrix**.

It follows:

$$\left(C_A^B\right)\left(C_A^B\right)^{-1} = C_A^B\left(C_A^B\right)^T = C_A^B\, C_B^A = C_B^B = I \tag{11-2}$$

Also,

$$\left(C_A^B\right)^{-1} C_A^B = \left(C_A^B\right)^T C_A^B = C_B^A\, C_A^B = C_A^A = I \tag{11-3}$$

In this Section, we will show that Equation (11-1) is valid for any arbitrary orientation of one frame relative to another frame, regardless of how frames may have reached the mutual orientation.

Consider two frames, Frame A and Frame B. Frame A has X_A, Y_A, and Z_A axes with unit vectors **I**, **J**, and **K**, respectively. Frame B has X_B, Y_B, and Z_B axes with unit vectors **i**, **j**, and **k**, respectively. An arbitrary vector has components x_A, y_A, and z_A in Frame A, and components x_B, y_B, and z_B in Frame B as shown in Figure 11-1.

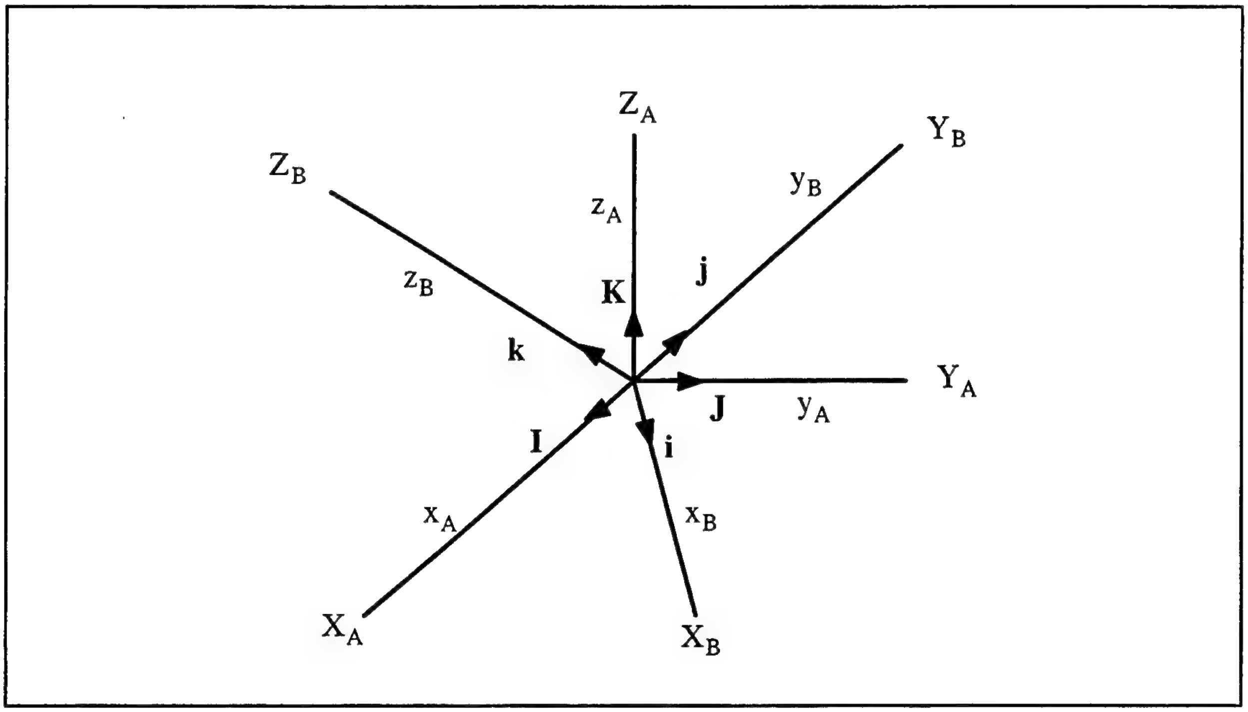


Figure 11-1. An Arbitrary Vector with Components Expressed in Frames A and B

We want to determine x_B, y_B and z_B of Frame B in terms of x_A, y_A, and z_A of Frame A. First consider x_B, which consists of,

$$
\begin{aligned}
x_B = &\ (\text{component of } x_A \text{ along } \mathbf{i} \text{ or } X_B \text{ axis}) \\
&+ (\text{component of } y_A \text{ along } \mathbf{i} \text{ or } X_B \text{ axis}) \\
&+ (\text{component of } z_A \text{ along } \mathbf{i} \text{ or } X_B \text{ axis})
\end{aligned}
$$

$$x_B = x_A \cos(\mathbf{i}, \mathbf{I}) + y_A \cos(\mathbf{i}, \mathbf{J}) + z_A \cos(\mathbf{i}, \mathbf{K}) \tag{11-4}$$

in which cos $(\mathbf{i}, \mathbf{I})$ is the cosine of angle formed by unit vector **i** and **I**, and so on.

Similarly,

$$y_B = x_A \cos(\mathbf{j}, \mathbf{I}) + y_A \cos(\mathbf{j}, \mathbf{J}) + z_A \cos(\mathbf{j}, \mathbf{K}) \tag{11-5}$$

$$z_B = x_A \cos(\mathbf{k}, \mathbf{I}) + y_A \cos(\mathbf{k}, \mathbf{J}) + z_A \cos(\mathbf{k}, \mathbf{K}) \tag{11-6}$$

From Equation (11-4) to Equation (11-6)

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} \cos(\mathbf{i},\mathbf{I}) & \cos(\mathbf{i},\mathbf{J}) & \cos(\mathbf{i},\mathbf{K}) \\ \cos(\mathbf{j},\mathbf{I}) & \cos(\mathbf{j},\mathbf{J}) & \cos(\mathbf{j},\mathbf{K}) \\ \cos(\mathbf{k},\mathbf{I}) & \cos(\mathbf{k},\mathbf{J}) & \cos(\mathbf{k},\mathbf{K}) \end{bmatrix}_A^B \begin{Bmatrix} x_A \\ y_A \\ z_A \end{Bmatrix} \tag{11-7}$$

The **Coefficient Matrix** of Equation (11-7) above is called **Direction Cosine Matrix** (which is also called **Rotation Matrix** or **Coordinate Transformation Matrix**) because all of its elements are cosines of the angle between directions of axes of two different frames.

By definition as we discussed in Section 1,

$$\mathbf{i} \cdot \mathbf{I} = |\mathbf{i}|\ |\mathbf{I}|\ \cos\ (\mathbf{i},\mathbf{I}) = \cos\ (\mathbf{i},\mathbf{I})$$

$$\mathbf{i} \cdot \mathbf{J} = |\mathbf{i}|\ |\mathbf{J}|\ \cos\ (\mathbf{i},\mathbf{J}) = \cos\ (\mathbf{i},\mathbf{J}), \text{etc.}$$

since **i**, **j**, **k** and **I**, **J**, **K** are unit vectors.

It follows that Equation (11-7) may be written as

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} \mathbf{i}\cdot\mathbf{I} & \mathbf{i}\cdot\mathbf{J} & \mathbf{i}\cdot\mathbf{K} \\ \mathbf{j}\cdot\mathbf{I} & \mathbf{j}\cdot\mathbf{J} & \mathbf{j}\cdot\mathbf{K} \\ \mathbf{k}\cdot\mathbf{I} & \mathbf{k}\cdot\mathbf{J} & \mathbf{k}\cdot\mathbf{K} \end{bmatrix}_A^B \begin{Bmatrix} x_A \\ y_A \\ z_A \end{Bmatrix} \tag{11-8}$$

or

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}_A^B \begin{Bmatrix} x_A \\ y_A \\ z_A \end{Bmatrix} \tag{11-9}$$

in which the meaning of C_{11}, C_{12}, etc. are obvious by comparing Equation (11-7) and Equation (11-8) to Equation (11-9).

We may summarize Equation (11-8) and Equation (11-9) by

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = C_A^B \begin{Bmatrix} x_A \\ y_A \\ z_A \end{Bmatrix}$$

or

$$\mathbf{r}^B = C_A^B\ \mathbf{r}^A \tag{11-10}$$

where C_A^B represents the **Coefficient Matrix** on the right side of Equation (11-9).

Next, we reverse the process. We want to find x_A, y_A, z_A in terms of x_B, y_B, z_B.

$$\begin{aligned} x_A = &\ (\text{component of } x_B \text{ along } \mathbf{I} \text{ or } X_A \text{ axis}) \\ &+ (\text{component of } y_B \text{ along } \mathbf{I} \text{ or } X_A \text{ axis}) \\ &+ (\text{component of } z_B \text{ along } \mathbf{I} \text{ or } X_A \text{ axis}) \end{aligned} \tag{11-11}$$

$$x_A = \cos\,(\mathbf{i},\mathbf{I})\ x_B + \cos\,(\mathbf{j},\mathbf{I})\ y_B + \cos\,(\mathbf{k},\mathbf{I})\ z_B \tag{11-12}$$

Similarly,

$$y_A = \cos\,(\mathbf{i},\mathbf{J})\ x_B + \cos\,(\mathbf{j},\mathbf{J})\ y_B + \cos\,(\mathbf{k},\mathbf{J})\ z_B \tag{11-13}$$

$$z_A = \cos\,(\mathbf{i},\mathbf{K})\ x_B + \cos\,(\mathbf{j},\mathbf{K})\ y_B + \cos\,(\mathbf{k},\mathbf{K})\ z_B \tag{11-14}$$

From Equation (11-12), Equation (11-13), and Equation (11-14):

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} \cos(\mathbf{i},\mathbf{I}) & \cos(\mathbf{j},\mathbf{I}) & \cos(\mathbf{k},\mathbf{I}) \\ \cos(\mathbf{i},\mathbf{J}) & \cos(\mathbf{j},\mathbf{J}) & \cos(\mathbf{k},\mathbf{J}) \\ \cos(\mathbf{i},\mathbf{K}) & \cos(\mathbf{j},\mathbf{K}) & \cos(\mathbf{k},\mathbf{K}) \end{bmatrix}_B^A \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \tag{11-15}$$

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} \mathbf{i}\cdot\mathbf{I} & \mathbf{j}\cdot\mathbf{I} & \mathbf{k}\cdot\mathbf{I} \\ \mathbf{i}\cdot\mathbf{J} & \mathbf{j}\cdot\mathbf{J} & \mathbf{k}\cdot\mathbf{J} \\ \mathbf{i}\cdot\mathbf{K} & \mathbf{j}\cdot\mathbf{K} & \mathbf{k}\cdot\mathbf{K} \end{bmatrix}_B^A \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \tag{11-16}$$

It follows

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = C_B^A \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$$

or

$$\mathbf{r}^A = C_B^A\ \mathbf{r}^B \tag{11-17}$$

where C_B^A represents the **Coefficient Matrix** on the right sides of Equation (11-15) and Equation (11-16).

Comparing C_B^A given in (11-15), with C_A^B given in (11-8), we note that C_B^A is equal to the Transpose (rows and columns exchanged) of C_A^B. That is,

$$C_B^A = (C_A^B)^T \text{ and } (C_A^B) = (C_B^A)^T. \tag{11-18}$$

Substituting Equation (11-10) into Equation (11-17):

$$\mathbf{r}^A = C_B^A\ \mathbf{r}^B$$

$$= C_B^A\ C_A^B\ \mathbf{r}^A. \tag{11-19}$$

Since $\mathbf{r}^A = I\ \mathbf{r}^A$ where I = **Identity Matrix**, we conclude:

$$C_B^A\ C_A^B = I. \tag{11-20}$$

Using Equation (11-18) in Equation (11-20):

$$(C_A^B)^T\ (C_A^B) = I. \tag{11-21}$$

Since $$(C_A^B)^{-1}\ (C_A^B) = I, \tag{11-22}$$

we conclude:

$$(C_A^B)^{-1} = (C_A^B)^T\ . \tag{11-23}$$

Since Equation (11-23) is true for any two **Orthogonal Rotation Matrixes**, we have, dropping subscripts and superscripts: $C^{-1} = C^T$.

It follows:

$$C^{-1}C = C^TC = I \tag{11-24}$$

$$CC^{-1} = CC^T = I. \tag{11-25}$$

Although Equation (11-18) through Equation (11-23) were shown in Section 5 using a single axis rotation, we are confirming them here for general case.

$$\text{Thus, for } C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{11-26}$$

we have, using Equation (11-24):

$$C^TC = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11-27}$$

Carrying out matrix multiplication:

$$C^TC = \begin{bmatrix} C_{11}^2 + C_{21}^2 + C_{31}^2 & C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32} & C_{11}C_{13} + C_{21}C_{23} + C_{31}C_{33} \\ C_{12}C_{11} + C_{22}C_{21} + C_{32}C_{31} & C_{12}^2 + C_{22}^2 + C_{32}^2 & C_{12}C_{13} + C_{22}C_{23} + C_{32}C_{33} \\ C_{13}C_{11} + C_{23}C_{21} + C_{33}C_{31} & C_{13}C_{12} + C_{23}C_{22} + C_{33}C_{32} & C_{13}^2 + C_{23}^2 + C_{33}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11-28}$$

Equating (1,1) elements:

$$C_{11}^2 + C_{21}^2 + C_{31}^2 = 1 \tag{11-29}$$

Equating (2,2) elements:

$$C_{12}^2 + C_{22}^2 + C_{32}^2 = 1 \tag{11-30}$$

Equating (3,3) elements:

$$C_{13}^2 + C_{23}^2 + C_{33}^2 = 1 \tag{11-31}$$

Equating (1,2) elements:

$$C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32} = 0 \tag{11-32}$$

Equating (1,3) elements:

$$C_{11}C_{13} + C_{21}C_{23} + C_{31}C_{33} = 0 \tag{11-33}$$

Equating (2,3) elements

$$C_{12}C_{13} + C_{22}C_{23} + C_{32}C_{33} = 0 \tag{11-34}$$

Referring to the matrix of C^TC given in Equation (11-28), we note that its (2,1) element is identical to its (1,2) element; (3,2) element to (2,3) element; and (3,1) element to (1,3) element.

If we use Equation (11-25), we have

$$CC^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \tag{11-35}$$

which leads to the following relationships among the coefficients, sometimes called the **redundancy relationships**.

It follows from Equation (11-35):

$$C_{11}^2 + C_{12}^2 + C_{13}^2 = 1 \tag{11-36}$$

$$C_{21}^2 + C_{22}^2 + C_{23}^2 = 1 \tag{11-37}$$

$$C_{31}^2 + C_{32}^2 + C_{33}^2 = 1 \tag{11-38}$$

Equating (1, 2) elements:

$$C_{11}C_{21} + C_{12}C_{22} + C_{13}C_{23} = 0 \tag{11-39}$$

Equating (2, 3) elements:

$$C_{21}C_{31} + C_{22}C_{32} + C_{23}C_{33} = 0 \tag{11-40}$$

Equating (1, 3) elements:

$$C_{11}C_{31} + C_{12}C_{32} + C_{13}C_{33} = 0 \tag{11-41}$$

Equation (11-29) through Equation (11-34) or, equivalently, Equation (11-36) through Equation (11-41) are results obtained from Equation (11-28) or Equation (11-35), repeated below:

$$C^TC = CC^T = I \tag{11-42}$$

which is called **orthogonality condition**.

When the **Rotation Matrices** are obtained experimentally or determined computationally with computation errors, they generally do not satisfy Equation (11-42). In those cases, Equation (11-29)

through Equation (11-34) or Equation (11-36) through Equation (11-41) are used to make the Rotation Matrices satisfy Equation (11-42). That is, the equations "normalize" and "orthogonalize" the Rotation Matrices.

The 3x3 **Direction Cosine Matrix** C has nine direction cosines. Six orthogonality relations — Equation (11-29) through Equation (11-34) or Equation (11-38) through Equation (11-41), — contain all nine elements. Therefore, these six equations may be solved for six elements in terms of three remaining elements. This means there are only three independent parameters. Since we cannot reduce the number of the independent parameters to less than three, making the number 3 the minimum number of independent parameters to specify the rotation, a set of three coordinate is called **generalized coordinates**. Although there are a number of such sets of parameters available, the most widely accepted are Euler Angles.

A more useful set of relationship than those given by Equations (11-39) through (11-41) follows from the orthogonality of the unit triads (**i, j, k**) and (**I, J, K**). By definition,

$$\mathbf{i} = \mathbf{j} \times \mathbf{k}; \quad \mathbf{j} = \mathbf{k} \times \mathbf{i}; \quad \mathbf{k} = \mathbf{i} \times \mathbf{j} \tag{11-43}$$

Applying Equation (11-9) between (**i, j, k**) and (**I, J, K**),

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix} \tag{11-44}$$

It follows from Equation (11-44):

$$\begin{aligned} \mathbf{i} &= C_{11}\mathbf{I} + C_{12}\mathbf{J} + C_{13}\mathbf{K} \\ \mathbf{j} &= C_{21}\mathbf{I} + C_{22}\mathbf{J} + C_{23}\mathbf{K} \\ \mathbf{k} &= C_{31}\mathbf{I} + C_{32}\mathbf{J} + C_{33}\mathbf{K} \end{aligned} \tag{11-45}$$

Using Equation (11-45) in Equation (11-43) for $\mathbf{i} = \mathbf{j} \times \mathbf{k}$:

$$\begin{aligned} & C_{11}\,\mathbf{I} + C_{12}\,\mathbf{J} + C_{13}\,\mathbf{K} \\ &= \left(C_{21}\,\mathbf{I} + C_{22}\,\mathbf{J} + C_{23}\,\mathbf{K}\right) \times \left(C_{31}\,\mathbf{I} + C_{32}\,\mathbf{J} + C_{33}\,\mathbf{K}\right) \\ &= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \left(C_{22}\,C_{33} - C_{32}\,C_{23}\right)\mathbf{I} \\ &+ \left(C_{23}\,C_{31} - C_{21}\,C_{33}\right)\mathbf{J} + \left(C_{21}\,C_{33} - C_{22}\,C_{31}\right)\mathbf{K} \end{aligned} \tag{11-46}$$

Equating coefficients for **I, J, K** on both sides of Equation (11-46), we get:

$$
\begin{aligned}
C_{11} &= C_{22}C_{33} - C_{32}C_{23} \\
C_{12} &= C_{23}C_{31} - C_{21}C_{33} \\
C_{13} &= C_{21}C_{32} - C_{31}C_{22}.
\end{aligned}
\tag{11-47}
$$

Similarly we obtain from $\mathbf{j} = \mathbf{k} \times \mathbf{i}$:

$$
\begin{aligned}
C_{21} &= C_{32}C_{13} - C_{12}C_{33} \\
C_{22} &= C_{11}C_{33} - C_{31}C_{13} \\
C_{23} &= C_{31}C_{12} - C_{11}C_{32}.
\end{aligned}
\tag{11-48}
$$

and from $\mathbf{k} = \mathbf{i} \times \mathbf{j}$:

$$
\begin{aligned}
C_{31} &= C_{12}C_{23} - C_{22}C_{13} \\
C_{32} &= C_{21}C_{13} - C_{11}C_{23} \\
C_{33} &= C_{11}C_{22} - C_{21}C_{12}.
\end{aligned}
\tag{11-49}
$$

Now going back to Equation (11-32) and solving for C_{11}:

$$
C_{11} = \frac{-\left(C_{21}C_{22} + C_{31}C_{32}\right)}{C_{12}}.
\tag{11-50}
$$

If C_{12} is 0, C_{11} cannot be obtained from Equation (11-50). However, it can be obtained from the first equation (Equation (11-47)) without these restrictions. In this sense, Equations (11-47), (11-48), and (11-49) are more useful than Equations (11-32), (11-33), and (11-34) because they allow us to avoid singularities.

SECTION 12

NORMALIZATION OF VECTOR AND ROTATION MATRIX

12.1 NORMALIZATION OF VECTORS

Given a vector **r**,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{12-1}$$

we want to normalize **r** so that its magnitude $|r| = 1$

That is, we require

$$|\mathbf{r}| = |\mathbf{i}x + \mathbf{j}y + \mathbf{k}z|$$

$$= (x^2 + y^2 + z^2)^{1/2} = 1 \tag{12-2}$$

Now define x', y', z', and $\mathbf{r}'$ by:

$$x' = \frac{x}{|\mathbf{r}|} \; ; \; y' = \frac{y}{|\mathbf{r}|} \; ; \; z' = \frac{z}{|\mathbf{r}|}$$

$$\mathbf{r}' = \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z' \tag{12-3}$$

Then

$$(x')^2 + (y')^2 + (z')^2 = \frac{x^2}{|\mathbf{r}|^2} + \frac{y^2}{|\mathbf{r}|^2} + \frac{z^2}{|\mathbf{r}|^2}$$

$$= \frac{x^2 + y^2 + z^2}{|\mathbf{r}|^2}$$

$$= \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$$

It follows that

$$|\mathbf{r}'| = \left[(x')^2 + (y')^2 + (z')^2\right]^{\frac{1}{2}} = 1 \tag{12-4}$$

which implies that $\mathbf{r}'$ is normalized.

12.2 NORMALIZATION OF MATRICES

For an **Orthogonal Matrix**:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \qquad (12\text{-}5)$$

such that

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = R\,,$$

we know if Equation (12-5) is errorless:

$$CC^T = I \quad \text{and} \quad C^TC = I \qquad (12\text{-}6)$$

or

$$CC^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad (12\text{-}7)$$

It follows from the above equation:

$$C_{11}^2 + C_{12}^2 + C_{13}^2 = 1$$

$$C_{21}^2 + C_{22}^2 + C_{23}^2 = 1$$

$$C_{31}^2 + C_{32}^2 + C_{33}^2 = 1 \qquad (12\text{-}8)$$

Equation (12-8) is called **Normalization condition**.

In some cases, a **Rotation Matrix** may not obey Equation (12-6). The next paragraphs show how to ensure that each diagonal element of CC^T is equal to I.

Now, assume the C_{ij}'s in Equation (12-5) have errors.

We want to normalize the C matrix so that Equation (12-8) is satisfied.

Now define C'_{R1}, C'_{R2} and C'_{R3} by:

$$C_{R1} \triangleq \left[C_{11}^2 + C_{12}^2 + C_{13}^2\right]^{1/2}$$

$$C_{R2} \triangleq \left[C_{21}^2 + C_{22}^2 + C_{23}^2\right]^{1/2}$$

$$C_{R3} \triangleq \left[C_{31}^2 + C_{32}^2 + C_{33}^2\right]^{1/2} \qquad (12\text{-}9)$$

and also define C'_{11}, C'_{12} and C'_{13} by:

$$C'_{11} \triangleq \frac{C_{11}}{C_{R1}}$$

$$C'_{12} \triangleq \frac{C_{12}}{C_{R1}}$$

$$C'_{13} \triangleq \frac{C_{13}}{C_{R1}} \qquad (12\text{-}10)$$

where the symbol $\triangleq$ denotes the definition.

It follows:

$$[C'_{11}]^2 + [C'_{12}]^2 + [C'_{13}]^2 = \frac{C_{11}^2 + C_{12}^2 + C_{13}^2}{C_{R1}{}^2} = \frac{C_{11}^2 + C_{12}^2 + C_{13}^2}{C_{11}^2 + C_{12}^2 + C_{13}^2} = 1 \qquad (12\text{-}11)$$

Define C'_{21}, C'_{22} and C'_{23} by:

$$C'_{21} \triangleq \frac{C_{21}}{C_{R2}}$$

$$C'_{22} \triangleq \frac{C_{22}}{C_{R2}}$$

$$C'_{23} \triangleq \frac{C_{23}}{C_{R2}} \qquad (12\text{-}12)$$

It follows:

$$[C'_{21}]^2 + [C'_{22}]^2 + [C'_{23}]^2 = \frac{C_{21}^2 + C_{22}^2 + C_{23}^2}{[C_{R2}]^2}$$

$$= \frac{C_{21}^2 + C_{22}^2 + C_{23}^2}{C_{21}^2 + C_{22}^2 + C_{23}^2} = 1 \tag{12-13}$$

Finally, define C'_{31}, C'_{32} and C'_{33} by:

$$C'_{31} = \frac{C_{31}}{C_{R3}}; \quad C'_{32} = \frac{C_{32}}{C_{R3}}; \quad C'_{33} = \frac{C_{33}}{C_{R3}} \tag{12-14}$$

It follows:

$$[C'_{31}]^2 + [C'_{32}]^2 + [C'_{33}]^2 = \frac{C_{31}^2 + C_{32}^2 + C_{33}^2}{[C_{R3}]^2}$$

$$= \frac{C_{31}^2 + C_{32}^2 + C_{33}^2}{C_{31}^2 + C_{32}^2 + C_{33}^2} = 1 \tag{12-15}$$

Therefore, the new matrix C' given by:

$$C' = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ C'_{21} & C'_{22} & C'_{23} \\ C'_{31} & C'_{32} & C'_{33} \end{bmatrix} \tag{12-16}$$

will satisfy Equation (12-8) with C_{ij} replaced by C'_{ij}, which means C' is normalized.

SECTION 13

ORTHOGONALIZATION OF VECTOR AND ROTATION MATRIX

13.1 ORTHOGONALIZATION OF VECTORS

Given a set of three non–orthogonal vectors with a common origin, we want to find a set of three mutually orthogonal vectors to form an orthogonal frame of reference.

That is, from three non–orthogonal axes $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, we want to find three orthogonal axes $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$. Figure 13-1 shows $\mathbf{x}_1$ and $\mathbf{x}_2$, which are not perpendicular to each other.

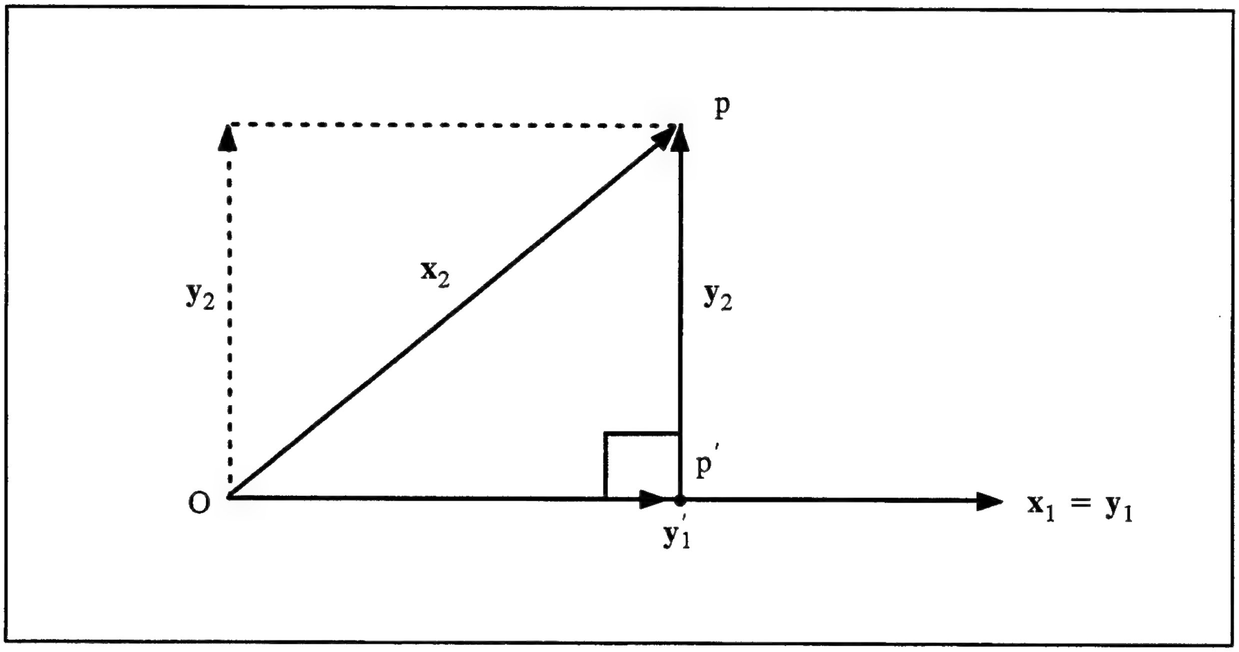


Figure 13-1. Two Non–Orthogonal Vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Initially we choose $\mathbf{y}_1$ to be $\mathbf{x}_1$, or $\mathbf{x}_1 = \mathbf{y}_1$. Then we draw a perpendicular line from p, (the tip of $\mathbf{x}_2$), to the $\mathbf{x}_1$ line, and call the intersection of this line with $\mathbf{x}_1$ line p'. Let $\mathrm{Op}' = \mathbf{y}_1'$, and $\mathrm{p}'\mathrm{p} = \mathbf{y}_2$.

Define k_1 by

$$\mathbf{y}_1' = k_1 \mathbf{y}_1$$

where k_1 is a scaling factor for $\mathbf{y}_1$.

It follows from Figure 13-1, using vector addition:

$$\mathbf{x}_2 = \mathbf{y}_1' + \mathbf{y}_2 = k_1\mathbf{y}_1 + \mathbf{y}_2 \tag{13-1}$$

From Equation (13-1):

$$\mathbf{y}_2 = \mathbf{x}_2 - k_1\mathbf{y}_1 \tag{13-2}$$

Since $\mathbf{y}_1$ is perpendicular to $\mathbf{y}_2$, we require that $\mathbf{y}_1 \cdot \mathbf{y}_2 = 0$
or, using Equation (13-2):

$$\mathbf{y}_1 \cdot \mathbf{y}_2 = \mathbf{y}_1 \cdot (\mathbf{x}_2 - k_1\mathbf{y}_1) = 0 \tag{13-3}$$

It follows:

$$\mathbf{y}_1 \cdot \mathbf{y}_2 = \mathbf{y}_1 \cdot \mathbf{x}_2 - k_1\mathbf{y}_1 \cdot \mathbf{y}_1 = 0 \tag{13-4}$$

which results in:

$$k_1 = \frac{\mathbf{y}_1 \cdot \mathbf{x}_2}{\mathbf{y}_1 \cdot \mathbf{y}_1} \tag{13-5}$$

Substituting Equation (13-5) into Equation (13-2) for k_1:

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\mathbf{y}_1 \cdot \mathbf{x}_2}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 \tag{13-6}$$

As a check to see if $\mathbf{y_1}$ is perpendicular to $\mathbf{y_2}$:

$$\mathbf{y}_1 \cdot \mathbf{y}_2 = \mathbf{y}_1 \cdot \left(\mathbf{x}_2 - \frac{\mathbf{y}_1 \cdot \mathbf{x}_2}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1\right) \tag{13-7}$$

$$= \mathbf{y}_1 \cdot \mathbf{x}_2 - \frac{\mathbf{y}_1 \cdot \mathbf{x}_2}{\mathbf{y}_1 \cdot \mathbf{y}_1}\mathbf{y}_1 \cdot \mathbf{y}_1$$

$$= \mathbf{y}_1 \cdot \mathbf{x}_2 - \mathbf{y}_1 \cdot \mathbf{x}_2 = 0 \tag{13-8}$$

which shows that $\mathbf{y}_2$ as determined by Equation (13-6) is indeed perpendicular to $\mathbf{y}_1$. Now that we found $\mathbf{y}_2$ determined by Equation (13-6) is perpendicular to $\mathbf{y}_1$, our next task is to find $\mathbf{y}_3$, which is perpendicular to both $\mathbf{y}_1$ and $\mathbf{y}_2$.

Referring to Figure 13-2, draw a perpendicular line from Q (the tip of $\mathbf{x}_3$) to the $\mathbf{y}_1 - \mathbf{y}_2$ plane, and call the intersection Q'. Drop a perpendicular line from Q' to Oy_1 line and call the intersection y_1''. Finally, draw a perpendicular line from Q' to Oy_2 line and call the intersection y_2''.

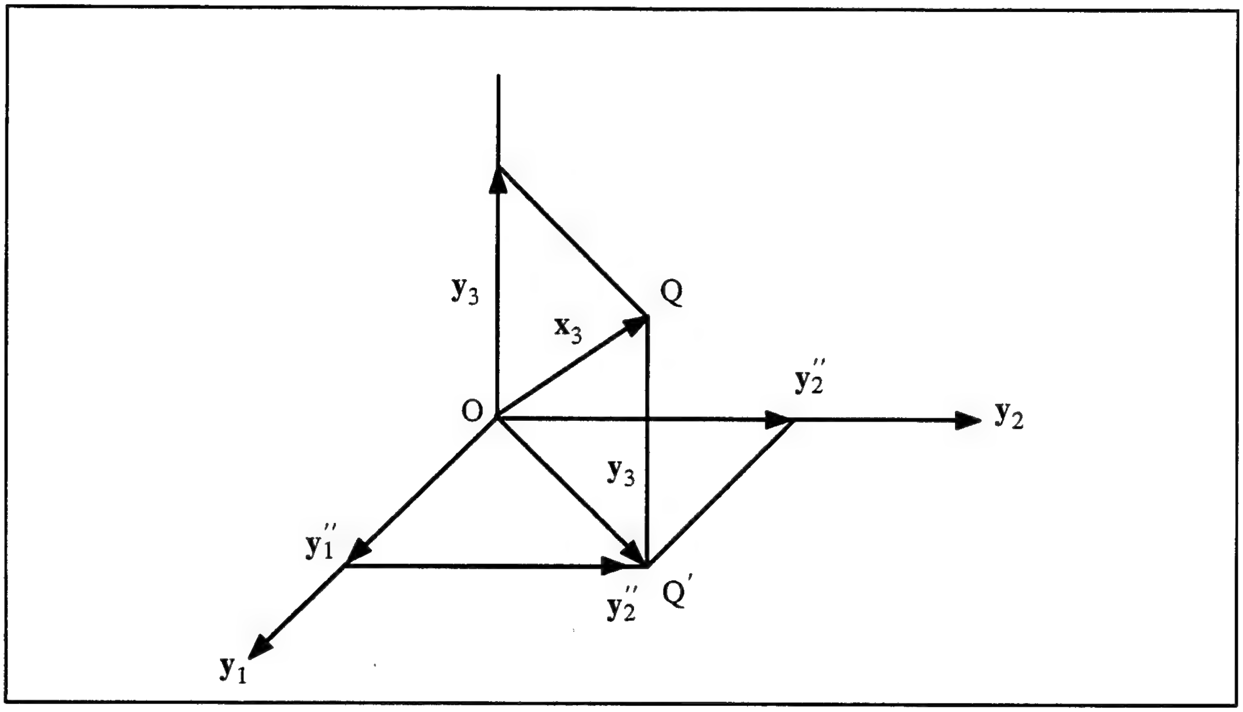


Figure 13-2. Orthogonal Set of Vectors

We define k_2 and k_3 by

$$\mathbf{y}_1'' = k_2\mathbf{y}_1 \tag{13-9}$$

and

$$\mathbf{y}_2'' = k_3\mathbf{y}_2 \tag{13-10}$$

where k_2 and k_3 are scaling factors for $\mathbf{y}_1$ and $\mathbf{y}_2$ respectively.

By simple vector addition:

$$OQ' = \mathbf{y}_1'' + \mathbf{y}_2'' = k_2\mathbf{y}_1 + k_3\mathbf{y}_2. \tag{13-11}$$

It follows:

$$\mathbf{x}_3 = OQ' + Q'Q = k_2\mathbf{y}_1 + k_3\mathbf{y}_2 + \mathbf{y}_3. \tag{13-12}$$

From Equation (13-12):

$$\mathbf{y}_3 = \mathbf{x}_3 - k_2\mathbf{y}_1 - k_3\mathbf{y}_2. \tag{13-13}$$

We want $\mathbf{y}_2$ and $\mathbf{y}_3$ to be perpendicular to each other, or $\mathbf{y}_2 \cdot \mathbf{y}_3 = 0$, or using Equation (13-13):

$$\mathbf{y}_2 \cdot \mathbf{y}_3 = \mathbf{y}_2 \cdot (\mathbf{x}_3 - k_2\mathbf{y}_1 - k_3\mathbf{y}_2)$$

$$= \mathbf{y}_2 \cdot \mathbf{x}_3 - k_2\mathbf{y}_2 \cdot \mathbf{y}_1 - k_3\mathbf{y}_2 \cdot \mathbf{y}_2 = 0 \quad (13\text{-}14)$$

Since we have shown previously that $\mathbf{y}_2 \cdot \mathbf{y}_1 = 0$, it follows from Equation (13-14):

$$k_3\mathbf{y}_2 \cdot \mathbf{y}_2 = \mathbf{y}_2 \cdot \mathbf{x}_3 \quad (13\text{-}15)$$

or

$$k_3 = \frac{\mathbf{y}_2 \cdot \mathbf{x}_3}{\mathbf{y}_2 \cdot \mathbf{y}_2} \quad (13\text{-}16)$$

Finally, we want $\mathbf{y}_1 \cdot \mathbf{y}_3 = 0$.

Using Equation (13-13) for $\mathbf{y}_3$:

$$\mathbf{y}_1 \cdot \mathbf{y}_3 = \mathbf{y}_1 \cdot (\mathbf{x}_3 - k_2\mathbf{y}_1 - k_3\mathbf{y}_2) \quad (13\text{-}17)$$

$$= \mathbf{y}_1 \cdot \mathbf{x}_3 - k_2\mathbf{y}_1 \cdot \mathbf{y}_1 - k_3\mathbf{y}_1 \cdot \mathbf{y}_2$$

$$= 0 \quad (13\text{-}18)$$

Again, since we shown previously that $\mathbf{y}_1 \cdot \mathbf{y}_2 = 0$, we get from Equation (13-18):

$$\mathbf{y}_1 \cdot \mathbf{x}_3 - k_2\mathbf{y}_1 \cdot \mathbf{y}_1 = 0 \quad (13\text{-}19)$$

or

$$k_2 = \frac{\mathbf{y}_1 \cdot \mathbf{x}_3}{\mathbf{y}_1 \cdot \mathbf{y}_1} \quad (13\text{-}20)$$

As a check to see if $\mathbf{y}_3$ is perpendicular to $\mathbf{y}_1$, using Equation (13-13) for $\mathbf{y}_3$ and Equation (13-20) for k_2:

$$\begin{aligned}
\mathbf{y}_1 \cdot \mathbf{y}_3 &= \mathbf{y}_1 \cdot (\mathbf{x}_3 - k_2\mathbf{y}_1 - k_3\mathbf{y}_2) \\
&= \mathbf{y}_1 \cdot \mathbf{x}_3 - k_2\mathbf{y}_1 \cdot \mathbf{y}_1 - k_3\mathbf{y}_1 \cdot \mathbf{y}_2 \\
&= \mathbf{y}_1 \cdot \mathbf{x}_3 - k_2\mathbf{y}_1 \cdot \mathbf{y}_1 \\
&= \mathbf{y}_1 \cdot \mathbf{x}_3 - \frac{\mathbf{y}_1 \cdot \mathbf{x}_3}{\mathbf{y}_1 \cdot \mathbf{y}_1}\, \mathbf{y}_1 \cdot \mathbf{y}_1 \\
&= \mathbf{y}_1 \cdot \mathbf{x}_3 - \mathbf{y}_1 \cdot \mathbf{x}_3 = 0 \qquad (13\text{-}21)
\end{aligned}$$

as expected because $\mathbf{y}_1 \cdot \mathbf{y}_2 = 0$.

To see if $\mathbf{y}_3$ is perpendicular to $\mathbf{y}_2$ using Equation (13-13) for $\mathbf{y}_3$ and Equation (13-16) for k_3:

$$\begin{aligned}
\mathbf{y}_2 \cdot \mathbf{y}_3 &= \mathbf{y}_2 \cdot (\mathbf{x}_3 - k_2\mathbf{y}_1 - k_3\mathbf{y}_2) \\
&= \mathbf{y}_2 \cdot \mathbf{x}_3 - k_2\mathbf{y}_2 \cdot \mathbf{y}_1 - k_3\mathbf{y}_2 \cdot \mathbf{y}_2 \\
&= \mathbf{y}_2 \cdot \mathbf{x}_3 - k_3\mathbf{y}_2 \cdot \mathbf{y}_2 \\
&= \mathbf{y}_2 \cdot \mathbf{x}_3 - \frac{\mathbf{y}_2 \cdot \mathbf{x}_3}{\mathbf{y}_2 \cdot \mathbf{y}_2}\, \mathbf{y}_2 \cdot \mathbf{y}_2 \\
&= \mathbf{y}_2 \cdot \mathbf{x}_3 - \mathbf{y}_2 \cdot \mathbf{x}_3 = 0 \qquad (13\text{-}22)
\end{aligned}$$

as expected because $\mathbf{y}_2 \cdot \mathbf{y}_1 = \mathbf{y}_1 \cdot \mathbf{y}_2 = 0$.

In summary, given a set of non–orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, we can find the corresponding orthogonal set $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$. First, we let $\mathbf{y}_1 = \mathbf{x}_1$. Then

$$\begin{aligned}
\mathbf{y}_2 &= \mathbf{x}_2 - k_1\mathbf{y}_1 \\
&= \mathbf{x}_2 - \left(\frac{\mathbf{y}_1 \cdot \mathbf{x}_2}{\mathbf{y}_1 \cdot \mathbf{y}_1}\right)\mathbf{y}_1
\end{aligned}$$

$$\mathbf{y}_3 = \mathbf{x}_3 - k_2\mathbf{y}_1 - k_3\mathbf{y}_2$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{y}_1 \cdot \mathbf{x}_3}{\mathbf{y}_1 \cdot \mathbf{y}_1}\right)\mathbf{y}_1 - \left(\frac{\mathbf{y}_2 \cdot \mathbf{x}_3}{\mathbf{y}_2 \cdot \mathbf{y}_2}\right)\mathbf{y}_2$$

13.2 ORTHOGONALIZATION OF ROTATION MATRICES

Referring to Section 11, consider matrix C:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \left(C_1\, C_2\, C_3\right) \tag{13-23}$$

where

$$\text{column } 1: \quad C_1 = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$

$$\text{column } 2: \quad C_2 = \begin{bmatrix} C_{12} \\ C_{22} \\ C_{32} \end{bmatrix}$$

$$\text{column } 3: \quad C_3 = \begin{bmatrix} C_{13} \\ C_{23} \\ C_{33} \end{bmatrix} \tag{13-24}$$

It follows:

$$C_1^T = \left(C_{11}\, C_{21}\, C_{31}\right); \quad C_2^T = \left(C_{12}\, C_{22}\, C_{32}\right); \quad C_3^T = \left(C_{13}\, C_{23}\, C_{33}\right) \tag{13-25}$$

The orthogonality condition states:

$$C^T C = I \tag{13-26}$$

or

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13-27}$$

Using Equation (13-24) and Equation (13-25) in Equation (13-27):

$$C^T C = \begin{bmatrix} C_1^T \\ C_2^T \\ C_3^T \end{bmatrix} \left(C_1\, C_2\, C_3\right) \tag{13-28}$$

From Equation (13-28):

$$\begin{bmatrix} C_1^T C_1 & C_1^T C_2 & C_1^T C_3 \\ C_2^T C_1 & C_2^T C_2 & C_2^T C_3 \\ C_3^T C_1 & C_3^T C_2 & C_3^T C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13-29}$$

The matrix C is called **normal** when $C^T C$ produces unity along principal diagonal and **orthogonal** when the off–diagonal elements are 0. When both conditions occur, the matrix is called **ortho–normal**.

Equating the diagonal elements:

$$C_1^T C_1 = 1 \,; \quad C_2^T C_2 = 1 \,; \quad C_3^T C_3 = 1 \tag{13-30}$$

Using Equation (13-24) and Equation (13-25):

$$C_1^T C_1 = \left(C_{11} \;\; C_{21} \;\; C_{31} \right) \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$

$$= C_{11}^2 + C_{21}^2 + C_{31}^2 = 1 \tag{13-31}$$

$$C_2^T C_2 = \left(C_{12} \;\; C_{22} \;\; C_{32} \right) = \begin{bmatrix} C_{12} \\ C_{22} \\ C_{32} \end{bmatrix}$$

$$= C_{12}^2 + C_{22}^2 + C_{32}^2 = 1 \tag{13-32}$$

$$C_3^2 C_3 = \left(C_{13} \;\; C_{23} \;\; C_{33} \right) \begin{bmatrix} C_{13} \\ C_{23} \\ C_{33} \end{bmatrix}$$

$$= C_{13}^2 + C_{23}^2 + C_{33}^2 = 1 \tag{13-33}$$

As a result of round–off errors in computations, the C matrix becomes increasingly non–orthogonal. If we assume that these vectors are normalized by the method described previously in Section 12, we have:

$$|\mathbf{C}_1| = |\mathbf{C}_2| = |\mathbf{C}_3| = 1 \tag{13-34}$$

where referring to (13-34), we constructed a set of new vectors, $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ such that

$$\mathbf{C}_1 = \mathbf{i}C_{11} + \mathbf{j}C_{21} + \mathbf{k}C_{31}$$

$$\mathbf{C}_2 = \mathbf{i}C_{12} + \mathbf{j}C_{22} + \mathbf{k}C_{32}$$

$$\mathbf{C}_3 = \mathbf{i}C_{13} + \mathbf{j}C_{23} + \mathbf{k}C_{33}$$

and where, based on the above equations:

$$|\mathbf{C}_1| = \sqrt{C_{11}^2 + C_{21}^2 + C_{31}^2}$$

$$|\mathbf{C}_2| = \sqrt{C_{12}^2 + C_{22}^2 + C_{32}^2}$$

$$|\mathbf{C}_3| = \sqrt{C_{13}^2 + C_{23}^2 + C_{33}^2}$$

As shown in Figure 13-3, $\mathbf{C}_1$ and $\mathbf{C}_2$ are not perpendicular to each other, at angle θ. If we want to make $\mathbf{C}_1$ and $\mathbf{C}_2$ to be perpendicular to each other, we will need to orthogonalize $\mathbf{C}_1$ and $\mathbf{C}_2$.

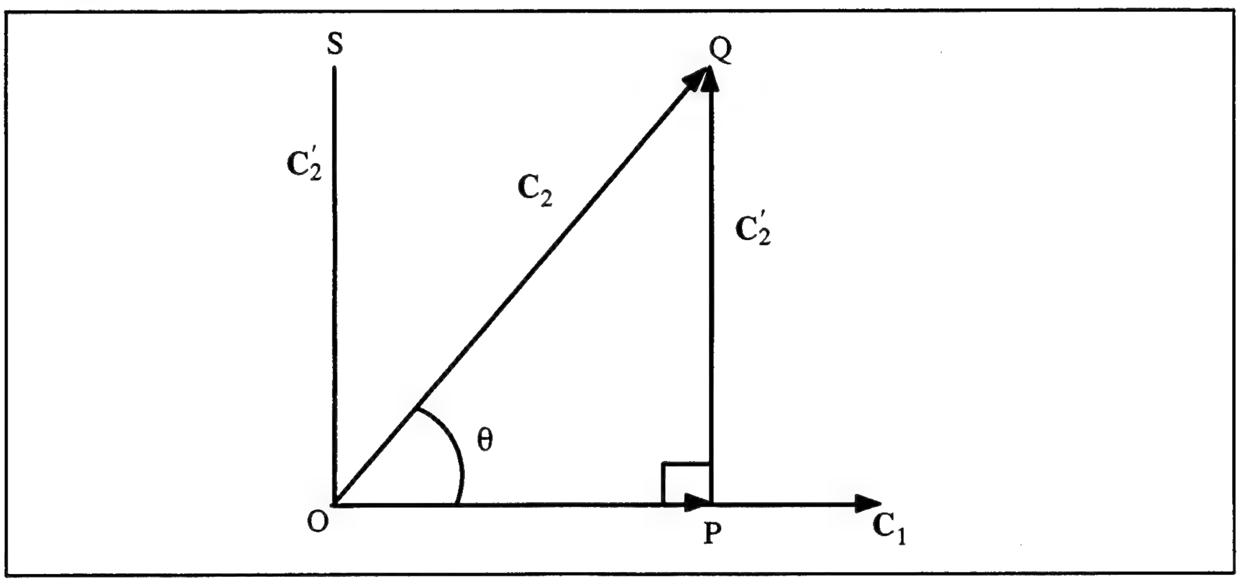


Figure 13-3. Non–Orthogonal Set of Column Vectors

Draw a perpendicular line from Q (the tip of $\mathbf{C}_2$) to $\mathbf{C}_1$, and call the intersection P.

By definition of the dot product:

$$\mathbf{C}_2^T\mathbf{C}_1 = \mathbf{C}_2 \cdot \mathbf{C}_1 = |\mathbf{C}_2|\,|\mathbf{C}_1| \cos\theta \tag{13-35}$$

The vector $\overline{\mathbf{OP}}$ is the component of $\mathbf{C}_2$ along the direction of $\mathbf{C}_1$. Denoting the unit vector along $\mathbf{C}_1$ by $\dfrac{\mathbf{C}_1}{|\mathbf{C}_1|}$, we have:

$$\begin{aligned}
\overline{\mathbf{OP}} &= |\mathbf{C}_2| \cos\theta \frac{\mathbf{C}_1}{|\mathbf{C}_1|} \\
&= |\mathbf{C}_2| \cos\theta \frac{\mathbf{C}_1}{|\mathbf{C}_1|}\frac{|\mathbf{C}_1|}{|\mathbf{C}_1|} \\
&= |\mathbf{C}_2|\,|\mathbf{C}_1| \cos\theta \frac{\mathbf{C}_1}{|\mathbf{C}_1|^2} \\
&= \mathbf{C}_2^T\,\mathbf{C}_1 \frac{\mathbf{C}_1}{|\mathbf{C}_1|^2}.
\end{aligned} \tag{13-36}$$

using (13-35).

Denote the vector $\overline{\mathbf{PQ}}$ by $\mathbf{C}_2'$.

Then, from Figure 13-3, by a vector addition,

$$\mathbf{C}_2 = \overline{\mathbf{OP}} + \mathbf{C}_2' \tag{13-37}$$

or

$$\begin{aligned}
\mathbf{C}_2' &= \mathbf{C}_2 - \overline{\mathbf{OP}} \\
&= \mathbf{C}_2 - (\mathbf{C}_2^T\mathbf{C}_1)\frac{\mathbf{C}_1}{|\mathbf{C}_1|^2}
\end{aligned}$$

by using Equation (13-36). (13-38)

Therefore, $\mathbf{C}_2'$ computed by Equation (13-38) using the known values of $\mathbf{C}_2$ and $\mathbf{C}_1$ is perpendicular to $\mathbf{C}_1$.

We started with $\mathbf{C}_1, \mathbf{C}_2,$ and $\mathbf{C}_3$, and made $\mathbf{C}_2'$ perpendicular to $\mathbf{C}_1$. The next step is to find $\mathbf{C}_3'$ (replacing $\mathbf{C}_3$), and make $\mathbf{C}_3'$ perpendicular to both $\mathbf{C}_1$ and $\mathbf{C}_2'$.

Define $\mathbf{C}_1''$ by $\mathbf{C}_1'' \triangleq a\mathbf{C}_1$ noting that $\mathbf{C}_1''$ is parallel to $\mathbf{C}_1$, and likewise define $\mathbf{C}_2''$ by $\mathbf{C}_2'' \triangleq b\mathbf{C}_2'$ noting that $\mathbf{C}_2''$ is parallel to $\mathbf{C}_2$, where a and b are scalar constants. (13-39)

Referring to Figure 13-4, and based on geometry:

$$\overline{\mathbf{OP}} = \mathbf{C}_1'' + \mathbf{C}_2''$$

$$= a\mathbf{C}_1 + b\mathbf{C}_2' \tag{13-40}$$

Also referring to Figure 13-4:

$$\mathbf{C}_3 = \overline{\mathbf{OP}} + \overline{\mathbf{PQ}}$$

$$= \left(\mathbf{C}_1'' + \mathbf{C}_2''\right) + \overline{\mathbf{PQ}}$$

$$= a\mathbf{C}_1 + b\mathbf{C}_2' + \mathbf{C}_3' \tag{13-41}$$

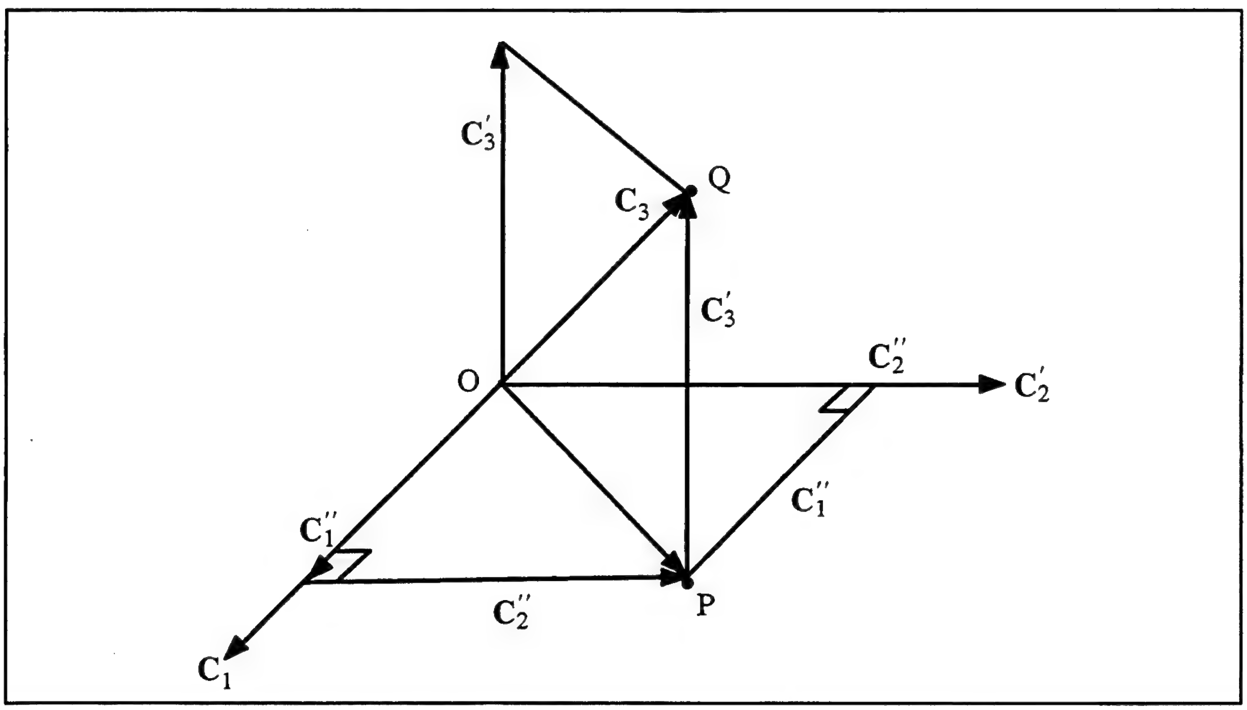


Figure 13-4. Defining Orthogonal Column Vectors

It follows from Equation (13-41):

$$\mathbf{C}_3' = \mathbf{C}_3 - a\mathbf{C}_1 - b\mathbf{C}_2' \tag{13-42}$$

Now $\mathbf{C}_3'$ is perpendicular to $\mathbf{C}_2'$.

That is,

$$\mathbf{C}_2' \cdot \mathbf{C}_3' = 0 \quad \text{or} \quad \left(\mathbf{C}_2'\right)^T \mathbf{C}_3' = 0. \tag{13-43}$$

Using Equation (13-42) in Equation (13-43):

$$\begin{aligned}\left(\mathbf{C}_2'\right)^T \mathbf{C}_3' &= \left(\mathbf{C}_2'\right)^T\left(\mathbf{C}_3 - a\mathbf{C}_1 - b\mathbf{C}_2'\right) \\ &= \left(\mathbf{C}_2'\right)^T \mathbf{C}_3 - a\left(\mathbf{C}_2'\right)^T \mathbf{C}_1 - b\left(\mathbf{C}_2'\right)^T \mathbf{C}_2' \\ &= \left(\mathbf{C}_2'\right)^T \mathbf{C}_3 - b\left(\mathbf{C}_2'\right)^T \mathbf{C}_2' \\ &= 0\end{aligned} \tag{13-44}$$

since $\left(\mathbf{C}_2'\right)^T \mathbf{C}_1 = 0$ because we previously made $\mathbf{C}_2'$ perpendicular to $\mathbf{C}_1$, so that

$$\mathbf{C}_2' \cdot \mathbf{C}_1 = \left(\mathbf{C}_2'\right)^T \mathbf{C}_1 = 0.$$

It follows from Equation (13-44):

$$b\left(\mathbf{C}_2'\right)^T \mathbf{C}_2' = \left(\mathbf{C}_2'\right)^T \mathbf{C}_3 \tag{13-45}$$

$$b = \frac{\left(\mathbf{C}_2'\right)^T \mathbf{C}_3}{\left(\mathbf{C}_2'\right)^T \mathbf{C}_2'} \tag{13-46}$$

Now, we want to make $\mathbf{C}_3'$ perpendicular to $\mathbf{C}_1$ (as well as perpendicular to $\mathbf{C}_2'$) or we require:

$$\mathbf{C}_1 \cdot \mathbf{C}_3' = 0 \quad \text{and} \quad \mathbf{C}_1^T \mathbf{C}_3' = 0 \tag{13-47}$$

Using Equation (13-42):

$$\mathbf{C}_1^T \mathbf{C}_3' = \mathbf{C}_1^T \left(\mathbf{C}_3 - a\mathbf{C}_1 - b\mathbf{C}_2'\right)$$

$$= \mathbf{C}_1^T \mathbf{C}_3 - a\mathbf{C}_1^T \mathbf{C}_1 - b\mathbf{C}_1^T \mathbf{C}_2'$$

$$= 0 \tag{13-48}$$

Since $\mathbf{C}_1$ is perpendicular to $\mathbf{C}_2'$, $\mathbf{C}_1^T \mathbf{C}_2' = 0$, we have from Equation (13-48):

$$a\,\mathbf{C}_1^T \mathbf{C}_1 = \mathbf{C}_1^T \mathbf{C}_3 \tag{13-49}$$

or

$$a = \frac{\mathbf{C}_1^T \mathbf{C}_3}{\mathbf{C}_1^T \mathbf{C}_1} \tag{13-50}$$

Substituting Equation (13-46) and Equation (13-50) into Equation (13-42):

$$\mathbf{C}_3' = \mathbf{C}_3 - a\mathbf{C}_1 - b\mathbf{C}_2'$$

$$= \mathbf{C}_3 - \frac{\mathbf{C}_1^T \mathbf{C}_3}{\mathbf{C}_1^T \mathbf{C}_1}\mathbf{C}_1 - \frac{\left(\mathbf{C}_2'\right)^T \mathbf{C}_3}{\left(\mathbf{C}_2'\right)^T \mathbf{C}_2'}\mathbf{C}_2' \tag{13-51}$$

Equation (13-51) may be evaluated in terms of known values of $\mathbf{C}_1$, $\mathbf{C}_3$, and $\mathbf{C}_2'$ given by Equation (13-38).

Next we show that $\mathbf{C}_3' \perp \mathbf{C}_2'$, using Equation (13-51) for $\mathbf{C}_3'$:

$$\left(\mathbf{C}_2'\right)^T \mathbf{C}_3' = \left(\mathbf{C}_2'\right)^T \left[\mathbf{C}_3 - \frac{\mathbf{C}_1^T \mathbf{C}_3}{\mathbf{C}_1^T \mathbf{C}_1}\mathbf{C}_1 - \frac{\left(\mathbf{C}_2'\right)^T \mathbf{C}_3}{\left(\mathbf{C}_2'\right)^T \mathbf{C}_2'}\mathbf{C}_2'\right] \tag{13-52}$$

It follows:

$$\left(\mathbf{C}_2'\right)^T \mathbf{C}_3' = \left(\mathbf{C}_2'\right)^T \mathbf{C}_3 - \frac{\mathbf{C}_1^T\ \mathbf{C}_3}{\mathbf{C}^T\ \mathbf{C}_1} \left(\mathbf{C}_2'\right)^T \mathbf{C}_1 - \frac{\left(\mathbf{C}_2'\right)^T \mathbf{C}_3}{\left(\mathbf{C}_2'\right)^T \mathbf{C}_2'} \left(\mathbf{C}_2'\right)^T \mathbf{C}_2'$$

$$= \left(\mathbf{C}_2'\right)^T \mathbf{C}_3 - \left(\mathbf{C}_2'\right)^T \mathbf{C}_3$$

$$= 0 \qquad (13\text{-}53)$$

because $\mathbf{C}_2' \perp \mathbf{C}_1$, and therefore $\left(\mathbf{C}_2'\right)^T \mathbf{C}_1 = 0$.

Equation (13-53) shows that $\mathbf{C}_2'$ is perpendicular to $\mathbf{C}_3'$.

It follows that $\mathbf{C}_1$, $\mathbf{C}_2'$, and $\mathbf{C}_3'$ (computed from given $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$) are mutually orthogonal.

The new orthogonal set $\mathbf{C}_1'$, $\mathbf{C}_2'$, and $\mathbf{C}_3'$ (denoting $\mathbf{C}_1$ by $\mathbf{C}_1'$), would be slightly off–normal because of the way it was derived. Thus we may have to re–normalize, which makes it slightly non–orthogonal. So, the process may have to be repeated until both normality and orthogonality criteria are met within error tolerance requirements. Usually only a few cycles will be sufficient.

SECTION 14

THEOREM OF CORIOLIS

Consider two frames, Frames A and B, with a common origin at 0. Frame B rotates with respect to Frame A with angular velocity $\mathbf{w}_{AB}$. A vector $\mathbf{R}_{op} = \mathbf{R}$ may be expressed in Frame A by

$$\mathbf{R}^A = \mathbf{I}X + \mathbf{J}Y + \mathbf{K}Z \tag{14-1}$$

and in Frame B by

$$\mathbf{R}^B = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z. \tag{14-2}$$

In the following equation, we use a shorthand notation P( ) for $\frac{d}{dt}$( ), and P_A for $\frac{d}{dt}$( ) with the differential increment observed in Frame A, and P_B for $\frac{d}{dt}$( ) with the differential increment observed in Frame B, and so on.

Now, taking $\frac{d}{dt}\mathbf{R}^A$ relative to Frame A, and denoting it by $P_A\mathbf{R}^A$:

$$\begin{aligned} P_A\mathbf{R}^A &= \mathbf{I}\frac{dX}{dt} + \mathbf{J}\frac{dY}{dt} + \mathbf{K}\frac{dZ}{dt} + X\frac{d\mathbf{I}}{dt} + Y\frac{d\mathbf{J}}{dt} + Z\frac{d\mathbf{K}}{dt} \\ &= \mathbf{I}\frac{dX}{dt} + \mathbf{J}\frac{dY}{dt} + \mathbf{K}\frac{dZ}{dt} \end{aligned} \tag{14-3}$$

since unit vectors **I**, **J**, and **K** are fixed in Frame A and do not vary with time in Frame A. Using Equation (14-2),

$$P_A\mathbf{R}^B = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt} + x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}. \tag{14-4}$$

In this case, the unit vectors **i**, **j**, and **k**, which are fixed in Frame B, rotate relative to Frame A, and therefore are time–varying if viewed from Frame A.

Now, consider $\frac{d\mathbf{i}}{dt}$ in Equation (14-4).

Referring to Figure 14-1, since vector **i** is a unit vector, it cannot change its magnitude. However, it can change its direction relative to Frame A because **i** is fixed in Frame B, which rotates relative to Frame A. For an infinitesimal angular displacement, the tip of the **i** vector moves on the

plane parallel to the $\mathbf{j}-\mathbf{k}$ plane. We can see this in the following way. If the frame is rotated infinitesimally about the **i**–axis, the direction of the **i**–vector is not changed.

If the frame is rotated infinitesimally about the z–axis, the tip of the **i**–vector moves (to the left) to the direction parallel to the **j**–axis. If the frame is rotated infinitesimally about the **j**–axis, the tip of the **i**–vector moves (downward) to the direction anti–parallel to the **k**–direction. Therefore, we may decompose $\Delta\mathbf{i}$ caused by the angular displacement of **i** of Frame B relative to Frame A in terms of its displacements in the **j** and **k** directions.

Referring to Figure 14-1, Frame A (with unit vectors **I**, **J**, **K**) coincides initially with Frame B (with unit vectors **i**, **j**, **k**). We see that the angular velocity $w_z = \frac{d\theta_z}{dt}$ about the Z–axis causes $\Delta\mathbf{i} = \mathbf{j}\Delta\theta_z = \mathbf{j}\frac{\Delta\theta_z}{\Delta t}\Delta t = \mathbf{j}w_z\Delta t$ during Δt, and the angular velocity $w_y = \frac{d\theta_y}{dt}$ in Y–axis causes $\Delta\mathbf{i} = -\mathbf{k}\,\theta_y = -\mathbf{k}\frac{\Delta\theta_y}{\Delta t}\Delta t = -\mathbf{k}\,w_y\,\Delta t$ during Δt. Summing the two components, we have:

$$\Delta\mathbf{i} = \mathbf{j}w_z\,\Delta t - \mathbf{k}w_y\,\Delta t \tag{14-5}$$

Dividing by Δt and taking the limits, we have

$$\frac{d\mathbf{i}}{dt} = \mathbf{j}w_z - \mathbf{k}w_y \tag{14-6}$$

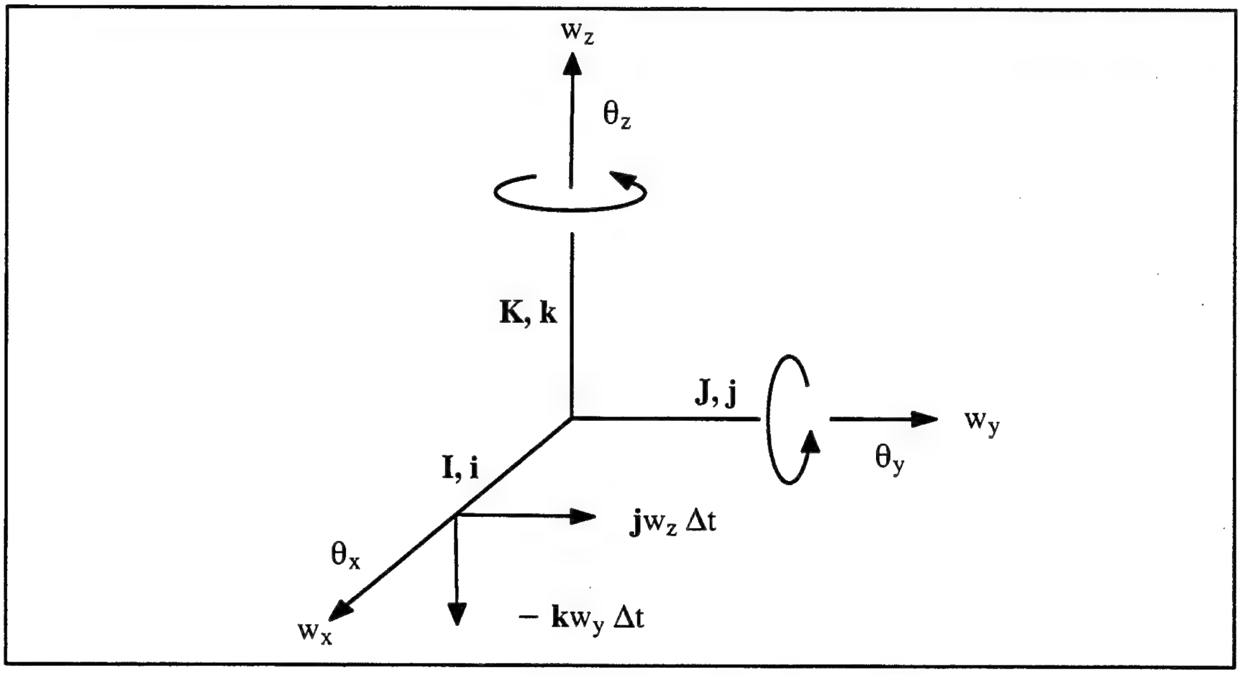


Figure 14-1. Incremental Rotation of Frame

The right side of Equation (14-6) is also equal to $\mathbf{w}^B \times \mathbf{i}$ as shown below, where the superscript B in $\mathbf{w}^B$ indicates that the components of $\mathbf{w}$ are expressed in Frame B. Since

$$\mathbf{w}^B = (w_x,\ \ w_y,\ \ w_z)^T \text{ and } \mathbf{i} = (1,\ 0,\ 0)^T\ ,$$

we have

$$\mathbf{w}^B \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_x & w_y & w_z \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}\, w_z - \mathbf{k}\, w_y \tag{14-7}$$

using the definition of the vector cross–product.

It follows that:

$$\frac{d\mathbf{i}}{dt} = \mathbf{w}^B \times \mathbf{i} = \mathbf{j}w_z - \mathbf{k}w_y \tag{14-8}$$

Similarly, we can show that:

$$\frac{d\mathbf{j}}{dt} = \mathbf{w}^B \times \mathbf{j} \tag{14-9}$$

$$\frac{d\mathbf{k}}{dt} = \mathbf{w}^B \times \mathbf{k} \tag{14-10}$$

Now, referring to the last three terms of the right side of Equation (14-4), and using Equations (14-8), (14-9), and (14-10):

$$x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt} = x(\mathbf{w}^B \times \mathbf{i}) + y(\mathbf{w}^B \times \mathbf{j}) + z(\mathbf{w}^B \times \mathbf{k})$$

$$= \mathbf{w}^B \times \mathbf{i}x + \mathbf{w}^B \times \mathbf{j}y + \mathbf{w}^B \times \mathbf{k}z \quad \text{(since x, y, z are}$$

scalars)

$$= \mathbf{w}^B \times (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$$

$$= \mathbf{w}^B \times \mathbf{R}^B \tag{14-11}$$

Note $\mathbf{w}^B$ is the angular velocity of Frame B relative to Frame A with components expressed in Frame B).

Now,

$$P_B\mathbf{R}^B = P_B(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$$

$$= \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt} \tag{14-12}$$

because **i**, **j**, **k** are fixed in Frame B, and therefore $\frac{d\mathbf{i}}{dt} = 0, \frac{d\mathbf{j}}{dt} = 0$ and $\frac{d\mathbf{k}}{dt} = 0$. Note that the right side of Equation (14-12) is equal to the first three terms of the right side of Equation (14-4).

So, substituting Equation (14-12) and Equation (14-11) into Equation (14-4):

$$P_A\mathbf{R}^B = P_B\mathbf{R}^B + \mathbf{w}^B_{AB} \times \mathbf{R}^B \tag{14-13}$$

In Equation (14-13), the vector $\mathbf{R}^B$ may be expressed, after appropriate transformations, in either Frame A or Frame B, (see Section 15). So, dropping the superscript B in Equation (14-13),

$$P_A\mathbf{R} = P_B\mathbf{R} + \mathbf{w}_{AB} \times \mathbf{R} \tag{14-14}$$

which is called the **Equation of Coriolis** or the **Theorem of Coriolis.**

What the Equation of Coriolis implies is that the differentiation (with respect to time) of a vector in one frame is not equal to the differentiation of the same vector in another frame that is rotating with respect to the first frame. And to obtain the value of $P_A\mathbf{R}$ in terms of $P_B\mathbf{R}$, we must add a correction term $\mathbf{w}_{AB} \times \mathbf{R}$ (which incorporates the effect of rotation of Frame B relative to Frame A) to $P_B\mathbf{R}$ as shown in Equation (14-14).

Readers who are interested in the geometrical approach in the derivation of Equation (14-14) may find it in other text books such as *"Mechanics"* by Keith R. Symon (published by Addison Wesley). Some may find the geometrical approach difficult to follow, while others may not. For this reason, an analytical approach is presented here to assist the comprehension in view of the importance of the theorem.

SECTION 15

MATRIX FORM OF THE THEOREM OF CORIOLIS

The Theorem of Coriolis in vector form (which we derived previously in Section 14) in which Frame B in rotating relative to Frame A with the angular velocity $\mathbf{w}_{AB}$ is given by:

$$P_A\mathbf{R} = P_B\mathbf{R} + \mathbf{w}_{AB} \times \mathbf{R}. \tag{15-1}$$

where $P_A\mathbf{R}$ means $\frac{d}{dt}\mathbf{R}$ observed in Frame A,

and $P_B\mathbf{R}$ means $\frac{d}{dt}\mathbf{R}$ observed in Frame B.

Expressing components of Equation (15-1) in Frame B:

$$\left[P_A\mathbf{R}\right]^B = \left[P_B\mathbf{R}\right]^B + \mathbf{w}_{AB}^B \times \mathbf{R}^B. \tag{15-2}$$

Now, referring to the left side of Equation (15-2) (using the notations previously explained in Section 5),

$$\begin{aligned}
\left[P_A\mathbf{R}\right]^B &= C_A^B \left[P_A\mathbf{R}\right]^A \\
&= C_A^B \left[\mathbf{P}\mathbf{R}^A\right] \\
&= C_A^B \left[P\left[C_B^A\mathbf{R}^B\right]\right] \quad \text{since} \quad \mathbf{R}^A = C_B^A\,\mathbf{R}^B \\
&= C_A^B \left[\left[P\,C_B^A\right]\mathbf{R}^B + C_B^A\left[\mathbf{P}\mathbf{R}^B\right]\right]
\end{aligned} \tag{15-3}$$

remembering that P( ) is the operator for $\frac{d}{dt}(\;)$, and recalling from calculus that the chain rule given below as:

$$\frac{d}{dt}[xy] = \left[\frac{d}{dt}x\right]y + x\frac{dy}{dt}$$

which is valid in the operation of Equation (15-3) as well.

It follows from Equation (15-3):

$$[P_A \mathbf{R}]^B = C_A^B \left[P C_B^A\right] \mathbf{R}^B + C_A^B \; C_B^A \; P\mathbf{R}^B$$

$$= C_A^B \left[P C_B^A\right] \mathbf{R}^B + P\mathbf{R}^B \tag{15-4}$$

since $C_A^B \, C_B^A = C_B^B = \mathbf{I}$.

Returning to Equation (15-2) and recalling that the vector $\mathbf{w}_{AB}^B \; \mathbf{XR}$ may be expressed in matrix form (as explained in Section 1) by:

$$\mathbf{w}_{AB}^B \times \mathbf{R}^B \Leftrightarrow \left[w_{AB}^{BK}\right] \mathbf{R}^B \tag{15-5}$$

where $\mathbf{w}_{AB}^B = \left(w_x w_y w_y\right)^T$ and

$$w_{AB}^{BK} = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

Substituting Equation (15-4) and Equation (15-5) into Equation (15-2),

$$C_A^B \left[P C_B^A\right] \mathbf{R}^B + P\mathbf{R}^B = P\mathbf{R}^B + \left[w_{AB}^{BK}\right] \mathbf{R}^B \tag{15-6}$$

noting that $(P_B \mathbf{R})^B = (P\mathbf{R}^B)^B = P\mathbf{R}^B$.

It follows that:

$$C_A^B \left[P C_B^A\right] \mathbf{R}_B = \left[w_{AB}^{BK}\right] \mathbf{R}^B. \tag{15-7}$$

Since Equation (15-7) has to be valid for all $\mathbf{R}^B$, we conclude:

$$C_A^B \; P\left[C_B^A\right] = w_{AB}^{BK} \tag{15-8}$$

Now, pre–multiplying (from the left) both sides of Equation (15-8) by $(C_A^B)^{-1} = (C_A^B)^T = C_B^A$ [because C_A^B is a Coordinate Transformation Matrix (between two orthogonal frames), and thus is an Orthogonal Matrix], we have:

$$C_B^A \; C_A^B \; P\left[C_B^A\right] = C_B^A \; w_{AB}^{BK} \tag{15-9}$$

Since $C_B^A C_A^B = C_A^A = \mathbf{I}$, from Equation (15-9):

$$PC_B^A = C_B^A \; w_{AB}^{BK}$$

$$\frac{d}{dt} C_B^A = C_B^A \; w_{AB}^{BK} \tag{15-10}$$

Equation (15-10) is the matrix form of the **Equation of Coriolis.**

Since

$$PC_B^A = \frac{d}{dt} C_B^A \simeq \frac{C_B^A(n+1) - C_B^A(n)}{\Delta t} \tag{15-11}$$

where C_B^A (n) denotes the value of C_B^A at n Δt, where Δt is the computation interval, we have from Equations (15-10) and (15-11):

$$C_B^A[n+1] = C_B^A[n] + C_B^A[n] \; w_{AB}^{BK}[n] \; \Delta t \tag{15-12}$$

Equation (15-12) suggests that we can update the orientation of the Eye frame (e.g., Frame B) relative to Head frame (e.g., Frame A) if we know the angular velocity $\mathbf{w}_{AB}$ which drives the eye ball.

w_{AB}^{BK} gives the angular velocity of Frame B (e.g., Eye frame) relative to Frame A (e.g., Head frame) with the components expressed in Frame B in a **skew–symmetric** form of the 3 x 3 matrix given in Equation (15-5) and repeated below:

$$w_{AB}^{BK} = \begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix} \tag{15-13}$$

The vector equivalent of Equation (15-13) is

$$\mathbf{w}_{AB}^{B} = \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix}^B \tag{15-14}$$

The angular velocity of B relative to A with components expressed in A–frame, $\mathbf{w}^{A}_{AB}$, may be obtained by a simple transformation:

$$\mathbf{w}^{A}_{AB} = \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix}^{A} = C^{A}_{B}\ \mathbf{w}^{B}_{AB}$$

$$= C^{A}_{B} \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix}^{B} \tag{15-15}$$

Then, the **Skew Symmetric Matrix** w^{AK}_{AB} may be constructed from $\mathbf{w}^{A}_{AB}$ obtained by Equation (15-15) using Equation (15-13).

Next, we want to derive the transformation equation to find w^{BK}_{BA} (with components resolved in Frame B) from w^{AK}_{BA} (with the components resolved in Frame A), and conversely to find w^{AK}_{AB} from w^{BK}_{AB}.

From Equation (15-10)

$$\frac{d}{dt} C^{A}_{B} = \dot{C}^{A}_{B} = C^{A}_{B}\ w^{BK}_{AB} \tag{15-16}$$

Since A and B are entirely arbitrary, exchange A and B in Equation (15-16):

$$\dot{C}^{B}_{A} = C^{B}_{A}\ w^{AK}_{BA} \tag{15-17}$$

Taking the transpose of both sides of Equation (15-17):

$$\left[\dot{C}^{B}_{A}\right]^{T} = \left[C^{B}_{A}\ w^{AK}_{BA}\right]^{T}$$

$$= \left[w^{AK}_{BA}\right]^{T} \left[C^{B}_{A}\right]^{T} \quad \text{(see Appendix A)} \tag{15-18}$$

Now, using Equation (15-13):

$$\left[w_{BA}^{AK}\right]^T = \begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & w_Z & -w_Y \\ -w_Z & 0 & w_X \\ w_Y & -w_X & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}$$

$$= -\, w_{BA}^{AK} \qquad (15\text{-}19)$$

Also, we know:

$$\left[C_A^B\right]^T = C_B^A \qquad (15\text{-}20)$$

Using Equation (15-19) and Equation (15-20) in Equation (15-18):

$$\dot{C}_B^A = -\left[w_{BA}^{AK}\right] C_B^A \qquad (15\text{-}21)$$

Now for the vector angular velocity:

$$\mathbf{w}_{AB} = -\,\mathbf{w}_{BA} \qquad (15\text{-}22)$$

It follows that:

$$w_{AB}^{AK} = -\,w_{BA}^{AK} \qquad (15\text{-}23)$$

Substituting Equation (15-23) into Equation (15-21):

$$\dot{C}_B^A = \left[w_{AB}^{AK}\right] C_B^A \qquad (15\text{-}24)$$

Since A and B are entirely arbitrary, exchanging A and B in Equation (15-24):

$$\dot{C}_A^B = w_{BA}^{BK}\, C_A^B \qquad (15\text{-}25)$$

Now,

$$C_A^B C_B^A = I \tag{15-26}$$

Since $\frac{d}{dt}I = 0$, differentiating Equation (15-26) using the chain rule:

$$\frac{d}{dt}\left[C_A^B C_B^A\right] = \dot{C}_A^B C_B^A + C_A^B \dot{C}_B^A$$

$$= 0 \tag{15-27}$$

Substituting Equation (15-25) and Equation (15-24) into Equation (15-27):

$$w_{BA}^{BK} C_A^B C_B^A + C_A^B w_{AB}^{AK} C_B^A = 0 \tag{15-28}$$

Using Equation (15-26) in Equation (15-28):

$$w_{BA}^{BK} = -C_A^B w_{AB}^{AK} C_B^A$$

$$= C_A^B\left[-w_{AB}^{AK}\right]C_B^A \tag{15-29}$$

Using Equation (15-23) in Equation (15-29):

$$w_{BA}^{BK} = C_A^B w_{BA}^{AK} C_B^A \tag{15-30}$$

which transforms w_{BA}^{AK} to w_{BA}^{BK}.

Since A and B are entirely arbitrary, exchange A and B in Equation (15-30):

$$w_{AB}^{AK} = C_B^A w_{AB}^{BK} C_A^B \tag{15-31}$$

which transforms w_{AB}^{BK} to w_{AB}^{AK}.

SECTION 16

QUATERNIONS

Quaternion has been an unmixed evil to those who have touched them in any way.

Lord Kelvin (1892)

Lord Kelvin's remark notwithstanding, quaternions, as far as their applications to eye movements are concerned, are useful.

The use of quaternions greatly facilitates the derivation of equations for the rotation angle and rotation vector in the Euler's Theorem of total equivalent rotation and the derivations of Rodrigues equation as indicated in the latter half of this report. Otherwise, the derivations of these equations seem to be almost too complicated to be tractable if we use conventional, non–quaternion algebra.

Historically, the quaternion is the result of the search for a "Three–Dimensional Complex Number." A complex number $z = x + \mathbf{i}y$ can represent a vector $\mathbf{r}$ in the plane. The complex numbers furnished the algebra for vectors.

But complex numbers are applicable only when all vectors lie on the same plane. To treat vectors in three–dimensional space, an analogue of complex numbers in three–dimensional space became necessary. The mathematicians in the first half of the nineteenth century searched for the three–dimensional complex numbers and associated algebra. This search led to the invention of quaternions by William Rowan Hamilton in 1843, which inspired the emergence of vector algebra in the latter part of the nineteenth century. It should be noted that, historically, quaternion algebra preceded vector algebra, not the other way around.

16.1 QUATERNION ALGEBRA

A quaternion q is a number of the form of

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \tag{16-1}$$

For example,

$$q = 2 + 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} \tag{16-2}$$

in which $\mathbf{i}, \mathbf{j}, \mathbf{k}$ play roles somewhat similar to $\mathbf{i}$ in the complex the number $z=a+\mathbf{i}b$. The real part q_0 of the quaternion, q, is called scalar part s(q), and $(\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)$ is called vector part $\mathbf{v}$(q). Thus Equation (16-1) is sometimes written as $q=s(q) + \mathbf{v}(q)$.

The conjugate of q denoted by q^* is obtained by making the vector part of q negative. That is,

$$q^* = q_0 - (\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)$$

$$= q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 \ \mathbf{k}q_3 \tag{16-3}$$

The three coefficients of the vector part are rectangular Cartesian coordinates of a point, while **i, j, k** are unit vectors along the three orthogonal axes.

The unit vectors **i, j, k** obey **Hamilton's rules** which are as follows :

$$\mathbf{ii} = \mathbf{i}^2 = \mathbf{jj} = \mathbf{j}^2 = \mathbf{kk} = \mathbf{k}^2 = -1 \tag{16-4}$$

(Note the similarity of the preceding rule to that of the conventional complex number, where $\mathbf{i} = \sqrt{-1}$ has the result $\mathbf{i}^2 = -1$.)

$$\mathbf{ij} = \mathbf{k} \qquad \mathbf{ji} = -\mathbf{k}$$

$$\mathbf{jk} = \mathbf{i} \qquad \mathbf{kj} = -\mathbf{i}$$

$$\mathbf{ki} = \mathbf{j} \qquad \mathbf{ik} = -\mathbf{j} \tag{16-5}$$

Note that the sign convention for Equation (16-5) is somewhat similar to that of vector cross–products, where, for example, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, etc.

Two quaternions p and q may be added or subtracted in a way similar to that of complex numbers. For:

$$p = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$$

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

We can find p+q and p–q simply:

$$p + q = (p_0 + q_0) + \mathbf{i}(p_1 + q_1) + \mathbf{j}(p_2 + q_2) + \mathbf{k}(p_3 + q_3)$$

$$p - q = (p_0 - q_0) + \mathbf{i}(p_1 - q_1) + \mathbf{j}(p_2 - q_2) + \mathbf{k}(p_3 - q_3) \tag{16-6}$$

Next, we want to find out if pq is equal to qp, that is, if the commutative law of multiplication holds. Now:

$$pq = (p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)\ (q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) \tag{16-7}$$

If we carry out the operations of Equation (16-7) using the rules of Equation (16-4) and Equation (16-5), we get:

$$\begin{aligned} pq = {} & (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) \\ & + \mathbf{i}(p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2) \\ & + \mathbf{j}(p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1) \\ & + \mathbf{k}(p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0) \end{aligned} \tag{16-8}$$

while

$$\begin{aligned} qp &= (q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)\,(p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}q_3) \\ &= (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) \\ & \quad + \mathbf{i}(q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2) \\ & \quad + \mathbf{j}(q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1) \\ & \quad + \mathbf{k}(q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0) \end{aligned} \tag{16-9}$$

Compare Equation (16-8) and Equation (16-9). Although real (scalar) parts are equal, not all the components of vector parts (**i**, **j**, **k** parts) have the same signs. We see that:

$$pq \neq qp \tag{16-10}$$

That is, the commutative law of multiplication does not hold for the quaternion multiplications, unlike the complex number multiplications.

16.2 QUATERNION OPERATION ON VECTORS

Quaternions may be used to rotate a vector as well as to change the length of a vector. Consider a vector $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. Now assume $\mathbf{r}$ is operated on by a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ from the right to become $\mathbf{r}' = \mathbf{i}x' + \mathbf{j}y' + \mathbf{h}z'$. That is,

$$q\mathbf{r} = \mathbf{r}' \tag{16-11}$$

or

$$(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)\,(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z' \tag{16-12}$$

After carrying out the operation of Equation (16-12), using Hamilton's Rule, we get

$$-(xq_1 - yq_3 - zq_3) + \mathbf{i}(xq_0 + zq_2 - yq_3)$$
$$+ \mathbf{j}(yq_0 - zq_1 + xq_3)$$
$$+ \mathbf{k}(zq_0 + yq_1 - xq_2)$$
$$= 0 + \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z' \tag{16-13}$$

Equating the coefficients of Equation (16-13) for real parts and **i**, **j** and **k** parts, we get:

$$xq_1 - yq_2 - zq_3 = 0 \tag{16-14}$$

$$xq_0 + zq_2 - yq_3 = x' \tag{16-15}$$

$$yq_0 - zq_1 + xq_3 = y' \tag{16-16}$$

$$zq_0 - yq_1 - xq_2 = z' \tag{16-17}$$

We now have four equations for four unknowns q_0, q_1, q_2, and q_3. These are the known values of the initial coordinates x, y, z of **r**, and the final coordinates x′, y′, z′ of **r**′, which is sufficient to solve for q_0, q_1, q_2 and q_3.

Thus if we want to rotate and elongate a given vector r (x, y, z) into a new vector **r**′(x′, y′, z′), we can do so mathematically, by the operation $q\mathbf{r} = \mathbf{r}'$ given by Equation (16-12) after first solving for q_0, q_1, q_2, q_3 using Equation (16-14).

16.3 LISTING'S LAW IN TERMS OF QUATERNION

The use of quaternion leads to a very simple formulation of Listing's Law (see Section 3).

A quaternion q may be written as:

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3. \tag{16-18}$$

The vector part $(\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)$ of q points in the direction of the axis of eye rotation. Listing's Law says that the axis must lie in the **j–k** plane (called equatorial plane). This simply means that the coordinate q_1 in **i**–direction is zero.

Thus, we can now state the effect of Listing's Law very simply: The torsion of the eye in any position is determined by a quaternion whose first vector component q_1 is zero, or by

$$q = q_0 + \mathbf{j}q_2 + \mathbf{k}q_3 \tag{16-19}$$

16.4 EXAMPLE OF QUATERNION ALGEBRA USING HAMILTON'S RULES

For two quaternions, v and q, denoted by

$$v = 0 + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \qquad \text{(with the scalar part equal to 0)} \tag{16-20}$$

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 ,$$

$$\begin{aligned} vq &= (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) \\ &= \mathbf{i}xq_0 + \mathbf{i}^2xq_1 + \mathbf{ij}xq_2 + \mathbf{ik}xq_3 \\ &\quad + \mathbf{j}yq_0 + \mathbf{ji}yq_1 + \mathbf{j}^2yq_2 + \mathbf{jk}yq_3 \\ &\quad + \mathbf{k}zq_0 + \mathbf{ki}zq_1 + \mathbf{kj}zq_2 + \mathbf{k}^2zq_3 \\ &= \mathbf{i}xq_0 - xq_1 + \mathbf{k}xq_2 - \mathbf{j}xq_3 \\ &\quad + \mathbf{j}yq_0 - \mathbf{k}yq_1 - yq_2 + \mathbf{i}yq_3 \\ &\quad + \mathbf{k}zq_0 + \mathbf{j}zq_1 - \mathbf{i}zq_2 - zq_3 \end{aligned} \tag{16-21}$$

It follows:

$$\begin{aligned} vq &= -(xq_1 + yq_2 + zq_3) \\ &\quad + \mathbf{i}(xq_0 + yq_3 - zq_2) \\ &\quad - \mathbf{j}(xq_3 - yq_0 - zq_1) \\ &\quad + \mathbf{k}(xq_2 - yq_1 + zq_0) \end{aligned} \tag{16-22}$$

Thus, vq generates a new quaternion denoted by $p = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$

where

$$\begin{aligned} p_0 &= -(xq_1 + yq_2 + zq_3) \\ p_1 &= (xq_0 + yq_3 - zq_2) \\ p_2 &= -(xq_3 - yq_0 - zq_1) \\ p_3 &= (xq_2 - yq_1 + zq_0) \end{aligned} \tag{16-23}$$

SECTION 17

EULER'S ROTATION VECTOR AND RELATIONSHIP BETWEEN QUATERNIONS AND ROTATION MATRICES

As we have shown in Section 16, a vector $\mathbf{r}$ operated on by q from the left resulting in $q\mathbf{r}$ which in turn generates another vector $\mathbf{r}'$ so that $q\mathbf{r} = \mathbf{r}'$.

Similarly, a vector $\mathbf{v} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ pre–multiplied by q^{-1} and post–multiplied by q generates an new vector, $\mathbf{v}' = \mathbf{i}x' + \mathbf{i}y' + \mathbf{k}z'$.

That is,

$$\mathbf{v}' = q^{-1}\,\mathbf{v}\,q \tag{17-1}$$

It follows:

$$(\mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z')$$

$$= \left(q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3\right)(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)\left(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\right) \tag{17-2}$$

In the rotation matrix notations:

$$\mathbf{v}' = C\,\mathbf{v} \tag{17-3}$$

or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{17-4}$$

where the matrix C here is used in the sense of Section 8, rather than Section 5.

The above equation is equivalent to:

$$\mathbf{i}\,x' = \mathbf{i}\left(C_{11}\,x + C_{12}\,y + C_{13}\right)$$

$$\mathbf{j}\,y' = \mathbf{j}\left(C_{21}\,x + C_{22}\,y + C_{23}\right)$$

$$\mathbf{k}\,z' = \mathbf{k}\left(C_{31}\,x + C_{32}\,y + C_{33}\right) \tag{17-5}$$

If we carry out the algebra of Equation (17-2) by using Hamilton's Rule (discussed in Section 16.1) we get using shorthand notations Q_{ij}:

$$\begin{aligned} \mathbf{v}' &= \mathbf{i}\,x' + \mathbf{j}\,y' + \mathbf{k}\,z' \\ &= \mathbf{i}\left(Q_{11}\,x + Q_{12}\,y + Q_{13}\,z\right) \\ &\quad + \mathbf{j}\left(Q_{21}\,x + Q_{22}\,y + Q_{23}\,z\right) \\ &\quad + \mathbf{k}\left(Q_{31}\,x + Q_{32}\,y + Q_{33}\,z\right) \end{aligned} \tag{17-6}$$

in which:

$$\begin{aligned} Q_{11} &= q_0^2 + q_1^2 - q_2^2 - q_3^2 = C_{11} \\ Q_{12} &= 2\left(q_0\,q_3 + q_1\,q_2\right) = C_{12} \\ Q_{13} &= 2\left(q_1\,q_3 - q_0\,q_2\right) = C_{13} \\ Q_{21} &= 2\left(q_1\,q_2 - q_0\,q_3\right) = C_{21} \\ Q_{22} &= q_0^2 - q_1^2 + q_2^2 - q_3^2 = C_{22} \\ Q_{23} &= 2\left(q_0\,q_1 + q_2\,q_3\right) = C_{23} \\ Q_{31} &= 2\left(q_0\,q_2 + q_1\,q_3\right) = C_{31} \\ Q_{32} &= 2\left(q_2\,q_3 - q_0\,q_1\right) = C_{32} \\ Q_{33} &= q_0^2 - q_1^2 - q_2^2 + q_3^2 = C_{32} \end{aligned} \tag{17-7}$$

Note that Equation (17-6) may also be written as:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{17-8}$$

Comparing Equation (17-8) with Equation (17-4):

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{17-9}$$

where Q_{ij} is given in Equation (17-7) in terms of the elements q_0, q_1, q_2, and q_3 of a quaternion q.

From Equation (17-7) and Equation (17-9):

$$\begin{aligned} C_{11} &= Q_{11} = q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ C_{22} &= Q_{22} = q_0^2 - q_1^2 + q_2^2 - q_3^2 \\ C_{33} &= Q_{33} = q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{aligned} \tag{17-10}$$

It follows from Equation (17-10):

$$\begin{aligned} C_{11} + C_{22} + C_{33} &= 3\,q_0^2 - q_1^2 - q_2^2 - q_3^2 \\ &= 3\,q_0^2 + q_0^2 - \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right) \\ &= 4\,q_0^2 - 1 \end{aligned} \tag{17-11}$$

because $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, (as derived in Equation (19-10)). It immediately follows from Equation (17-11):

$$q_0 = \frac{1}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}} \tag{17-12}$$

Also, from Equation (17-7) and Equation (17-9):

$$C_{23} - C_{32} = Q_{23} - Q_{32} = 4\, q_1\, q_0 \tag{17-13}$$

$$q_1 = \frac{C_{23} - C_{32}}{4\, q_0} \tag{17-14}$$

$$C_{31} - C_{13} = Q_{31} - Q_{13} = 4\, q_2\, q_0 \tag{17-15}$$

$$q_2 = \frac{C_{31} - C_{13}}{4\, q_0} \tag{17-16}$$

$$C_{12} - C_{21} = Q_{12} - Q_{21} = 4\, q_0\, q_3 \tag{17-17}$$

$$q_3 = \frac{C_{12} - C_{21}}{4\, q_0} \tag{17-18}$$

Thus, we now have expressions for q_0, q_1, q_2, and q_3 in terms of the elements C_{ij} of the rotation matrix with Equation (17-12) in which q_0 is given in terms of C_{11}, C_{22}, and C_{33}.

Conversely, if want to express C_{ij} in terms of q_0, q_1, q_2, and q_3, they are given in Equation (17-7).

Using Equation (17-12) through Equation (17-18), the vector part $\mathbf{v}(q)$ of the quaternion q may be expressed in terms of the elements C_{ij} of the rotation matrix as follows:

$$\mathbf{v}(q) = \mathbf{i}\,\frac{C_{23} - C_{32}}{4\, q_0} + \mathbf{j}\,\frac{C_{31} - C_{13}}{4\, q_0} + \mathbf{k}\,\frac{C_{12} - C_{21}}{4\, q_0} \tag{17-19}$$

where q_0 is given by Equation (17-12):

$$q_0 = \frac{1}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}}$$

Now, we want to express the rotation vector along the Euler axis in terms of the unit vector **u** and its components u_x, u_y, and u_z along X, Y, and Z axes. From Section 19, on "The Angle of Quaternion," we have:

$$u_1 = u_x = \frac{q_1}{\sin\frac{\phi}{2}}$$

$$u_2 = u_y = \frac{q_2}{\sin\frac{\phi}{2}}$$

$$u_3 = u_z = \frac{q_3}{\sin\frac{\phi}{2}} \qquad (17\text{-}20)$$

Using Equation (17-14), Equation (17-16), and Equation (17-18) in Equation (17-20):

$$u_x = \frac{1}{\sin\frac{\phi}{2}} \frac{C_{23} - C_{32}}{2\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}}}$$

$$u_y = \frac{1}{\sin\frac{\phi}{2}} \frac{C_{31} - C_{13}}{2\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}}}$$

$$u_z = \frac{1}{\sin\frac{\phi}{2}} \frac{C_{12} - C_{21}}{2\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}}} \qquad (17\text{-}21)$$

where $\frac{\phi}{2}$ is the one half of the rotation angle ϕ about the Euler axis.

Now, we want to express Equation (17-21) in terms of ϕ, not $\frac{\phi}{2}$, utilizing the identity shown below:

$$\sin\frac{\phi}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}} = \sin\phi \qquad (17\text{-}22)$$

which will be derived at the end of this section, and given in Equation (17-34).

Using Equation (17-22) in Equation (17-21):

$$u_x = \frac{C_{23} - C_{32}}{2 \sin\phi}$$

$$u_y = \frac{C_{31} - C_{13}}{2 \sin\phi}$$

$$u_z = \frac{C_{12} - C_{21}}{2 \sin\phi} \tag{17-23}$$

Equation (17-23) shows the components of the unit vector $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$ along the Euler axis in terms of C_{ij} of the rotation matrix and Euler's principal angle ϕ. Although it is relatively easy to derive here by means of quaternion algebra, it would have been extremely complicated, had we not used quaternions.

Next, we derive Equation (17-22).

From a standard mathematical table:

$$\cos 2\alpha = 2\cos^2\alpha - 1 \tag{17-24}$$

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha \tag{17-25}$$

From Equation (17-11):

$$C_{11} + C_{22} + C_{33} + 1 = 4 q_0^2 \tag{17-26}$$

Since $q_0 = \cos\frac{\phi}{2}$ by definition (see Section 19), from Equation (17-26):

$$C_{11} + C_{22} + C_{33} + 1 = 4 \cos^2 \frac{\phi}{2} \tag{17-27}$$

From Equation (17-24), replacing 2α by ϕ so that $\alpha = \frac{\phi}{2}$:

$$\cos\phi = 2 \cos^2 \frac{\phi}{2} - 1$$

$$\cos\phi + 1 = 2 \cos^2 \frac{\phi}{2}$$

$$2(\cos\phi + 1) = 4 \cos^2 \frac{\phi}{2} \tag{17-28}$$

Using Equation (17-28) in Equation (17-27):

$$C_{11} + C_{22} + C_{33} + 1 \;=\; 2\cos\phi + 2 \tag{17-29}$$

This gives an important equation:

$$C_{11} + C_{22} + C_{33} \;=\; 2\cos\phi + 1 \tag{17-30}$$

which says that the trace (sum of the diagonal elements of the rotation matrix) is equal to twice the cosine of the rotation angle about the the Euler axis plus 1.

Referring to the right side of Equation (17-29), and using the first equation of Equation (17-28) for $\cos\phi$:

$$\begin{aligned}(2 + 2\cos\phi)^{\frac{1}{2}} &= \left[2 + 2\left(2\cos^2\frac{\phi}{2} - 1\right)\right]^{\frac{1}{2}} \\ &= \left(2 + 4\cos^2\frac{\phi}{2} - 2\right)^{\frac{1}{2}} \\ &= \left(4\cos^2\frac{\phi}{2}\right)^{\frac{1}{2}} \\ &= 2\cos\frac{\phi}{2}\end{aligned} \tag{17-31}$$

ignoring the negative root.

Using Equation (17-31) in Equation (17-29):

$$\left(C_{11} + C_{22} + C_{33} + 1\right)^{\frac{1}{2}} \;=\; (2\cos\phi + 2)^{\frac{1}{2}} \;=\; 2\cos\frac{\phi}{2} \tag{17-32}$$

From Equation (17-25) replacing $2\,\alpha$ by ϕ so that $\alpha = \frac{\phi}{2}$:

$$\sin\phi \;=\; 2\,\sin\frac{\phi}{2}\cos\frac{\phi}{2} \tag{17-33}$$

Substituting Equation (17-32) into the left side of Equation (17-22), and using Equation (17-33) we have:

$$\sin\frac{\phi}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}} \;=\; \left(\sin\frac{\phi}{2}\right)2\left(\cos\frac{\phi}{2}\right) \;=\; \sin\phi \tag{17-34}$$

which is what we wanted to show.

SECTION 18

EULER'S ROTATION VECTOR RATE

As shown in Section 2, the Euler's Rotation vector **u** (u for unit vector) has the same direction cosines both in the Eye Frame and the Head Frame. That is,

$$\mathbf{u}^E = \mathbf{u}^H \tag{18-1}$$

It is also true that

$$\mathbf{u}^E = C_H^E \, \mathbf{u}^H \tag{18-2}$$

It follows:

$$C_H^E \, \mathbf{u}^H = \mathbf{u}^H \tag{18-3}$$

Differentiating Equation (18-3) with respect to time:

$$\frac{dC_H^E}{dt} \, \mathbf{u}^H + C_H^E \, \frac{d\mathbf{u}^H}{dt} = \frac{d\mathbf{u}^H}{dt} \tag{18-4}$$

or

$$\frac{dC_H^E}{dt} \, \mathbf{u}^H = \left(I - C_H^E\right) \frac{d\mathbf{u}^H}{dt} \tag{18-5}$$

Referring to Section 15:

$$\frac{d}{dt} \, C_B^A = C_B^A \, w_{AB}^{B*} \tag{18-6}$$

where we used * in place of k in superscript.

Identifying H with A and E with B,

$$\frac{d}{dt} C_E^H = C_E^H \, w_{HE}^{E*} \tag{18-7}$$

Taking the transpose of Equation (18-7):

$$\left[\frac{d}{dt} C_E^H\right]^T = \left[C_E^H \, w_{HE}^{E*}\right]^T$$

$$= \left[w_{HE}^{E*}\right]^T \left[C_E^H\right]^T \tag{18-8}$$

Now, $\left[C_E^H\right]^T = C_H^E$ for Orthogonal Matrix. Therefore,

$$\left[\frac{d}{dt} C_E^H\right]^T = \left[\frac{d}{dt} C_H^E\right] \tag{18-9}$$

and, referring to Section 1 and other sections, we have

$$w_{HE}^{E*} = \begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix} \tag{18-10}$$

Therefore,

$$\begin{aligned} \left[w_{HE}^{E*}\right]^T &= \begin{bmatrix} 0 & w_Z & -w_Y \\ -w_Z & 0 & w_X \\ w_Y & -w_X & 0 \end{bmatrix} \\ &= -\begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix} \qquad (18\text{-}11) \\ &= -\, w_{HE}^{E*} \end{aligned}$$

A matrix A is called **skew–symmetric** if $A = -A^T$ such as Equation (18-11).

Substituting Equation (18-9) and Equation (18-11) into Equation (18-8):

$$\frac{d}{dt} C_H^E = -\, w_{HE}^{E*}\, C_H^E \tag{18-12}$$

Substituting Equation (18-12) into Equation (18-5):

$$-\, w_{HE}^{E*}\, C_H^E\, \mathbf{u}^H = \left(I - C_H^E\right) \frac{d\mathbf{u}^H}{dt} \tag{18-13}$$

Substituting Equation (18-3) into Equation (18-13):

$$-\, w_{HE}^{E*}\, \mathbf{u}^H = \left(I - C_H^E\right) \frac{d\mathbf{u}^H}{dt} \tag{18-14}$$

As an intermediate step to find $\frac{du}{dt}$ in terms of ϕ and **u**, we want to find a 3 x 3 matrix M in terms of ϕ and **u** with the following characteristics:

$$1)\quad M^T M = I \tag{18-15}$$

$$2)\quad |M| = 1 \tag{18-16}$$

$$3)\quad \mathbf{Mu} = \mathbf{u} \tag{18-17}$$

$$4)\quad M_{11} + M_{22} + M_{33} = \text{trace } M$$

$$= 1 + 2\cos\phi \tag{18-18}$$

Equation (18-15) requires that M be an Orthogonal Matrix. Equation (18-16) follows from Equation (18-15). Equation (18-17) and Equation (18-18) complete the requirements for M to be identical to the Rotation Matrix C.

The following Equation (18-19) meets the above requirements from Equation (18-15) through Equation (18-18):

(See, for example, *Space Craft Attitude Dynamics* by Peter C. Hughes, John Wiley, 1986)

$$M \triangleq \cos\phi\, I + (1 - \cos\phi)\, \mathbf{uu}^T - \sin\phi\, u^* \tag{18-19}$$

where

$$\mathbf{uu}^T = \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} \begin{bmatrix} u_X & u_Y & u_Z \end{bmatrix}$$

$$= \begin{bmatrix} u_X u_X & u_X u_Y & u_X u_Z \\ u_Y u_X & u_Y u_Y & u_Y u_Z \\ u_Z u_X & u_Z u_Y & u_Z u_Z \end{bmatrix} \tag{18-20}$$

and u* is the skew–symmetric form of the vector $\mathbf{u} = [u_X\ u_Y\ u_Z]^T$, similar to Equation (18-10) for w_{HE}^{E*} in form, and given by:

$$u^* = \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} \tag{18-21}$$

Similar to Equation (18-11), it is obvious that

$$\left[u^*\right]^T = -u^*$$

Expanding Equation (18-19):

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \cos\phi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \ (1 - \cos\phi) \begin{bmatrix} u_X^2 & u_X\, u_Y & u_X\, u_Z \\ u_Y\, u_X & u_Y^2 & u_Y\, u_Z \\ u_Z\, u_X & u_Z\, u_Y & u_Z^2 \end{bmatrix}$$

$$- \sin\phi \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} \tag{18-22}$$

Expanding the right–hand side of Equation (18-22), and equating the element by element:

$$M_{11} = (1 - \cos\phi)\ u_1^2 + \cos\phi$$

$$M_{22} = (1 - \cos\phi)\ u_2^2 + \cos\phi$$

$$M_{33} = (1 - \cos\phi)\ u_3^2 + \cos\phi$$

$$M_{12} = (1 - \cos\phi)\ u_1 u_2 + u_3 \sin\phi$$

$$M_{21} = (1 - \cos\phi)\ u_2 u_1 - u_3 \sin\phi$$

$$M_{23} = (1 - \cos\phi)\ u_2 u_3 + u_1 \sin\phi$$

$$M_{32} = (1 - \cos\phi)\ u_3 u_2 - u_1 \sin\phi$$

$$M_{31} = (1 - \cos\phi)\ u_3 u_1 + u_2 \sin\phi$$

$$M_{13} = (1 - \cos\phi)\ u_1 u_3 - u_2 \sin\phi \tag{18-23}$$

Before we prove that M given by Equation (18-19) satisfies the requirements given by Equation (18-15) to Equation (18-18), we need several mathematical relationships, which are given below:

$$\mathbf{u}^T\mathbf{u} = \begin{bmatrix} u_X \, u_Y \, u_Z \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix}$$

$$= u_X^2 + u_Y^2 + u_Z^2 = 1 \tag{18-24}$$

$$\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T = \mathbf{u}\left(\mathbf{u}^T\mathbf{u}\right)\mathbf{u}^T = \mathbf{u}\mathbf{u}^T \tag{18-25}$$

$$u^*\mathbf{u} = \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix}$$

$$= \begin{bmatrix} -u_Z \, u_Y + u_Y \, u_Z \\ u_Z \, u_X - u_X \, u_Z \\ -u_Y \, u_X + u_X \, u_Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{18-26}$$

$$\mathbf{u}^T u^* = \begin{bmatrix} u_X \, u_Y \, u_Z \end{bmatrix} \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_Y \, u_Z - u_Z \, u_Y & -u_X \, u_Z + u_Z \, u_X & u_X \, u_Y - u_Y \, u_X \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{18-27}$$

$$u^* u^* = \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -u_Z^2 - u_Y^2 & u_Y\, u_X & -u_Z\, u_X \\ u_X\, y_Y & -u_Z^2 - u_X^2 & u_Z\, u_Y \\ u_X\, u_Z & u_Y\, u_Z & -u_Y^2 - u_X^2 \end{bmatrix} \tag{18-28}$$

Since for the Euler's unit matrix, $u_X^2 + u_Y^2 + u_Z^2 = 1$, we may express Identity Matrix I by

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_X^2 + u_Y^2 + u_Z^2 & 0 & 0 \\ 0 & u_X^2 + u_Y^2 + u_Z^2 & 0 \\ 0 & 0 & u_X^2 + u_Y^2 + u_Z^2 \end{bmatrix} \tag{18-29}$$

It follows from Equation (18-20), Equation (18-28), and Equation (18-29):

$$u^* u^* = \mathbf{u}\mathbf{u}^T - I \tag{18-30}$$

Now, from Equation (18-19):

$$M^T = \left[\cos\phi I + (1 - \cos\phi)\mathbf{u}\mathbf{u}^T - \sin\phi u^*\right]^T$$

$$= \cos\phi I^T + (1 - \cos\phi)\left[\mathbf{u}\mathbf{u}^T\right]^T - \sin\phi\left[u^*\right]^T$$

$$= \cos\phi I + (1 - \cos\phi)\mathbf{u}\mathbf{u}^T + \sin\phi u^* \tag{18-31}$$

since $I = I^T$, $\left[\mathbf{u}\mathbf{u}^T\right]^T = \left[\mathbf{u}^T\right]^T \mathbf{u}^T = \mathbf{u}\mathbf{u}^T$ and $\left[u^*\right]^T = -u^*$.

Let us determine if Equation (18-15) is satisfied. Using Equations (18-31) and (18-19):

$$M^TM = \left[\cos\phi I + (1 - \cos\phi)\mathbf{uu}^T + \sin\phi u^*\right] \left[\cos\phi I + (1 - \cos\phi)\mathbf{uu}^T - \sin\phi u^*\right]$$

$$= \cos^2\phi I + \cos\phi(1 - \cos\phi)\mathbf{uu}^T - \cos\phi\sin\phi u^*$$

$$+ (1 - \cos\phi)\cos\phi\mathbf{uu}^T + (1 - \cos\phi)^2\mathbf{u}\left(\mathbf{u}^T u\right)\mathbf{u}^T - (1 - \cos\phi)'\mathbf{u}\left(\mathbf{u}^T u^*\right)$$

$$+ \sin\phi\cos\phi u^* + \sin\phi(1 - \cos\phi)\left(u^*\mathbf{u}\right)\mathbf{u}^T - \sin^2\phi u^* u^* \tag{18-32}$$

After some algebra, Equation (18-32) reduces to:

$$M^TM = \cos^2\phi I + \sin^2\phi I$$

$$+ \text{[terms which cancel each other]}$$

$$+ \text{[terms which equal to zero]}$$

$$= I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18-33}$$

This proves Equation (18-15), establishing that M is an Orthogonal Matrix.

After Equation (18-15) is established, it is a simple matter to prove Equation (18-16). Taking the determinant of Equation (18-15), and using Equation (18-33):

$$|M^TM| = |M^T|\ |M|$$

$$= |M|\ |M| = |M|^2$$

$$= |I| = 1 \tag{18-34}$$

It follows from $|M|^2 = 1$,

$$|M| = \pm 1 \tag{18-35}$$

Referring to Equation (18-19), M is a continuous function of ϕ, and evaluating $|M|$ at $\phi = 0$:

$$M = |I + 0 - 0| = |I| = 1 \tag{18-36}$$

Therefore, we conclude

$$|M| = +1 \tag{18-37}$$

which proves Equation (18-16).

To prove Equation (18-17), we use direct multiplication of M given by Equation (18-19) by **u** and using Equation (18-24) and Equation (18-26):

$$\mathbf{Mu} = \left[\cos\phi I + (1 - \cos\phi)\,\mathbf{uu}^T - \sin\phi u^*\right]\mathbf{u}$$

$$= \left[\cos\phi\mathbf{u} + \mathbf{u}\left(\mathbf{u}^T\mathbf{u}\right) - \cos\phi\mathbf{u}\left(\mathbf{u}^T\mathbf{u}\right) - \sin\phi\left(u^*\mathbf{u}\right)\right]$$

$$= \cos\phi\mathbf{u} + \mathbf{u} - \cos\phi\mathbf{u} - 0$$

$$= \mathbf{u} \tag{18-38}$$

Finally, to prove Equation (18-18), we use the first three equations of Equation (18-23):

$$\text{Trace } M = M_{11} + M_{22} + M_{33}$$

$$= (1 - \cos\phi)\,u_1^2 + \cos\phi + (1 - \cos\phi)\,u_2^2 + \cos\phi$$

$$= +(1 - \cos\phi)\,u_3^2 + \cos\phi$$

$$= u_1^2 + u_2^2 + u_3^2 - \cos\phi\left(u_1^2 + u_2^2 + u_3^2\right) + 3\cos\phi$$

$$= 1 - \cos\phi + 3\cos\phi$$

$$= 1 + 2\cos\phi \tag{18-39}$$

since $u_1^2 + u_2^2 + u_3^2 = 1$ for unit vector.

In summary, we have confirmed that M is identical to the Rotation Matrix C_H^E, or

$$C_H^E = M = \cos\phi I + (1 - \cos\phi)\,\mathbf{uu}^T - \sin\phi u^* \tag{18-40}$$

Substituting Equation (18-40) into the right side of Equation (18-14) for C_H^E:

$$-\left[w_{HE}^{E*}\right] \mathbf{u} = \left\{I - \left[\mathbf{uu}^T - \cos\phi\left(I - \mathbf{uu}^T\right) - u * \sin\phi\right]\right\} \frac{d\mathbf{u}}{dt}$$

$$= \left[I - \mathbf{uu}^T + \cos\phi I - \cos\phi\, \mathbf{uu}^T + u^* \sin\phi\right] \frac{d\mathbf{u}}{dt}$$

$$= \frac{d\mathbf{u}}{dt} - \mathbf{u}\left(\mathbf{u}^T \frac{d\mathbf{u}}{dt}\right) + \cos\phi \frac{d\mathbf{u}}{dt} - \cos\phi \mathbf{u}\left(\mathbf{u}^T \frac{d\mathbf{u}}{dt}\right) + u^* \sin\phi \frac{d\mathbf{u}}{dt} \tag{18-41}$$

Now, $\mathbf{u}$ is a unit vector. Therefore, it cannot change its magnitude, but it can change its direction only. This means that the infinitesimal change represented by $\frac{d\mathbf{u}}{dt}$ must be at the perpendicular direction (right angle) from the direction of $\mathbf{u}$. That is,

$$\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = \mathbf{u}^T \frac{d\mathbf{u}}{dt} = 0 \tag{18-42}$$

since $\cos 90^0 = 0$.

Using Equation (18-42) in Equation (18-41):

$$-\left[w_{HE}^{E*}\right] \mathbf{u} = \frac{d\mathbf{u}}{dt} - \cos\phi \frac{d\mathbf{u}}{dt} + u^* \sin\phi \frac{d\mathbf{u}}{dt} \tag{18-43}$$

$$= \left[(1 - \cos\phi)I + u^* \sin\phi\right] \frac{d\mathbf{u}}{dt} \tag{18-44}$$

Multiplying both sides of Equation (18-44) by the inverse of $\left[(1 - \cos\phi)I + u^* \sin\phi\right]$ from the left:

$$\frac{d\mathbf{u}}{dt} = \left[(1 - \cos\phi)I + u^* \sin\phi\right]^{-1} \left[- w_{HE}^{E*} \mathbf{u}\right] \tag{18-45}$$

Now, using Equations (18-10), (18-11), and (18-21), we can show that:

$$- w_{HE}^{E*} \mathbf{u} = u^* w_{HE}^{E} \tag{18-46}$$

or

$$\begin{bmatrix} 0 & w_Z & -w_Y \\ -w_Z & 0 & w_X \\ w_Y & -w_X & 0 \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix}$$

$$= \begin{bmatrix} w_Z u_Y - w_Y u_Z \\ -w_Z u_X + w_X u_Z \\ w_Y u_X - w_X u_Y \end{bmatrix} \quad (18\text{-}47)$$

This confirms, for two vectors **a** and **b**, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ may be expressed $a^*\mathbf{b} = -b^*\mathbf{a}$ using a Skew–symmetric Matrix.

Using Equation (18-46) in Equation (18-45):

$$\frac{d\mathbf{u}}{dt} = \left[(1 - \cos\phi)I + u * \sin\phi\right]^{-1} u^* \mathbf{w}_{HE}^{E} \quad (18\text{-}48)$$

Note that $\mathbf{w}_{HE}^{E}$ is a 3 x 1 column vector, while w_{HE}^{E*} is a 3 x 3 mat rix equivalent of the vector $\mathbf{w}_{HE}^{E}$, which is the angular velocity of the eye relative to head with components expressed in the Eye frame.

The inverse indicated in Equation (18-48) may be expressed as

$$\left[(1 - \cos\phi)\, I + u^* \sin\phi\right]^{-1}$$

$$= \frac{1}{1 - \cos\phi} \left[I - \frac{1}{2} \sin\, \phi u^* + \frac{1}{2} (1 + \cos\phi) \left(\mathbf{u}\mathbf{u}^T - I\right)\right] \quad (18\text{-}49)$$

The validity of the right side of Equation (18-49) may be checked by multiplying it from the left by $\left[(1 - \cos\phi)\, I + u^* \sin\phi\right]$ and confirming that the result reduces to the Identity Matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Substituting Equation (18-49) into Equation (18-48):

$$\frac{d\mathbf{u}}{dt} = \frac{1}{1 - \cos\phi} \left\{u^* - \frac{1}{2} \sin\, \phi \left[u^* u^*\right] - \frac{1}{2} (1 + \cos\phi)\, u^*\right\} \mathbf{w}_{HE}^{E} \quad (18\text{-}50)$$

where we used the relation

$$\frac{1}{2}(1 + \cos\phi)\left(\mathbf{u}\mathbf{u}^T - I\right)u^* = \frac{1}{2}(1 + \cos\phi)\left[\mathbf{u}\left(\mathbf{u}^T u^*\right) - u^*\right]$$

$$= -\frac{1}{2}(1 + \cos\phi)\, u^* \qquad (18\text{-}51)$$

since $\mathbf{u}^T u^* = [0\ \ 0\ \ 0]$ by Equation (18-27).

It follows from Equation (18-50):

$$\frac{d\mathbf{u}}{dt} = \frac{1}{1 - \cos\phi}\left[\frac{1}{2}u^* - \frac{1}{2}\cos\phi u^* - \frac{1}{2}\sin\phi u^* u^*\right]\mathbf{w}_{HE}^{E}$$

$$= \frac{1}{1 - \cos\phi}\left[\frac{1}{2}(1 - \cos\phi)\, u^* - \frac{1}{2}\sin\phi\, u^* u^*\right]\mathbf{w}_{HE}^{E} \qquad (18\text{-}52)$$

From Equation (18-52):

$$\frac{d\mathbf{u}}{dt} = \left[\frac{1}{2}u^* - \frac{1}{2}\cot\frac{\phi}{2}\, u^* u^*\right]\mathbf{w}_{HE}^{E} \qquad (18\text{-}53)$$

in which we used a trigonometry identity:

$$\cot\frac{\phi}{2} = \frac{\sin\phi}{1 - \cos\phi} \qquad (18\text{-}54)$$

Using Equation (18-30) for u^*u^*, Equation (18-53) may also be written as

$$\frac{d\mathbf{u}}{dt} = \left[\frac{1}{2}u^* - \frac{1}{2}\cos\frac{\phi}{2}\left(\mathbf{u}\mathbf{u}^T - I\right)\right]\mathbf{w}_{HE}^{E} \qquad (18\text{-}55)$$

Equation (18-53) or Equation (18-55) is the time rate of the unit vector along the Euler's single equivalent rotation.

The equation for $\frac{d\mathbf{u}}{dt}$ as well as $\frac{d\phi}{dt}$ for the rotation angular rate (scalar) about the Euler's principal axis (see Section 22) is difficult to solve. One reason is that the vector **u** is defined only after a rotation has taken place; consequently, there is some problem in setting an initial value for **u**.

SECTION 19

THE ANGLE OF QUATERNION: THE ROTATION ANGLE OF EULER'S THEOREM

Consider a unit vector **u** along the axis of a single equivalent rotation in the context of the Euler's Theorem that makes angle α with both X_A and X_B axes, angle β with both Y_A and Y_B axes and angle γ with both Z_A and Z_B axes. Then, the components u_1, u_2 and u_3 of **u** along X, Y, and Z axes of both Frames A and B are related to α, β, and γ (see Section 2):

$$u_1 = \cos\alpha \quad \text{along X axis}$$

$$u_2 = \cos\beta \quad \text{along Y axis}$$

$$u_3 = \cos\gamma \quad \text{along Z axis.}$$

The quaternion q may be expressed as:

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 = q_0 + \mathbf{q} \tag{19-1}$$

in which

q_0 is called the scalar part of q, denoted by s(q), and

($\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$) is the vector part of q, denoted by **v**(q).

So, we may write:

$$q = s(q) + \mathbf{v}(q) \tag{19-2}$$

Assume Frame B (e.g., moving Eye frame) is rotated by angle ϕ from Frame A (reference–primary frame) about the Euler axis.

Euler four–parameters q_0, q_1, q_2 and q_3 (which obey the quaternion algebra explained in Section 16) are defined by:

$$q_0 = \cos\frac{1}{2}\phi \tag{19-3}$$

$$q_1 = u_1 \sin\frac{1}{2}\phi = \cos\alpha \sin\frac{1}{2}\phi \tag{19-4}$$

$$q_2 = u_2 \sin\frac{1}{2}\phi = \cos\beta \sin\frac{1}{2}\phi \tag{19-5}$$

$$q_3 = u_3 \sin\frac{1}{2}\phi = \cos\gamma \sin\frac{1}{2}\phi \tag{19-6}$$

Substituting Equation (19-3) through Equation (19-6) into Equation (19-1):

$$q = \cos\frac{1}{2}\phi + \mathbf{i}\cos\alpha\ \sin\frac{1}{2}\phi + \mathbf{j}\cos\beta\ \sin\frac{1}{2}\phi + \mathbf{k}\cos\gamma\ \sin\frac{1}{2}\phi$$

$$= \cos\frac{1}{2}\phi + (\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma)\ \sin\frac{1}{2}\phi \qquad (19\text{-}7)$$

Since

$$\mathbf{i}\ \cos\ \alpha + \mathbf{j}\ \cos\ \beta + \mathbf{k}\ \cos\ \gamma = \mathbf{i}\ u_1 + \mathbf{j}\ u_2 + \mathbf{k}\ u_3 = \mathbf{u} \qquad (19\text{-}8)$$

we have from Equation (19-7):

$$q = \cos\frac{\phi}{2} + \mathbf{u}\sin\frac{\phi}{2} \qquad (19\text{-}9)$$

in which ϕ (not $\frac{\phi}{2}$) is called the angle of the quaternion q and **u** its axis.

Using Equations (19-3), (19-4), (19-5), and (19-6):

$$q_0^2 + q_1^2 + q_2^2 + q_3^2$$

$$= \cos^2\frac{\phi}{2} + \left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)\sin^2\frac{\phi}{2}$$

$$= \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}$$

$$= 1 \qquad (19\text{-}10)$$

since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ based on Equation (19-8).

Adding the squares of Equation (19-4), (19-5), and (19-6):

$$q_1^2 + q_2^2 + q_3^2 = \left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)\sin^2\frac{\phi}{2}$$

$$= \sin^2\frac{\phi}{2} \qquad (19\text{-}11)$$

It follows:

$$\sin\frac{\phi}{2} = \left(q_1^2 + q_2^2 + q_3^2\right)^{1/2} \qquad (19\text{-}12)$$

If the Eye frame (Frame B) is rotated from the reference–primary orientation by angle ϕ about the Euler axis with unit vector **u**, then we represent the new orientation of the Eye frame by

$$q = \cos\frac{\phi}{2} + \mathbf{u}\sin\frac{\phi}{2} \tag{19-13}$$

Inverse of q : q^{-1}

Previously, we defined the conjugate of q^* of a quaternion q, given by

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \tag{19-14}$$

to be

$$q^* = q_0 - \left(\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\right) = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3 \tag{19-15}$$

It follows:

$$q\,q^* = \left(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\right)\left(q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3\right) \tag{19-16}$$

$$q^*q = \left(q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3\right)\left(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\right) \tag{19-17}$$

In both Equation (19-16) and Equation (19-17), if we carry out the algebra using the Hamilton's Rule explained in Section 16, we get the same result as follows:

$$\begin{aligned} q\,q^* &= q^*\,q = q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_0^2 + \left(q_1^2 + q_2^2 + q_3^2\right) \\ &= \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} = 1 \end{aligned} \tag{19-18}$$

using Equations (19-3) and (19-11).

By definition, the inverse denoted by q^{-1} of q must satisfy:

$$qq^{-1} = q^{-1}q = 1 \tag{19-19}$$

From Equation (19-18) and Equation (19-19), we conclude that:

$$q^{-1} = q^*$$

The inverse q^{-1} of q is equal to the conjugate q^* of q if we use the definition given in Equations (19-3), (19-4), (19-5), and (19-6). That is,

$$q^{-1} = q^* = q_0 - \left(\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\right) \tag{19-20}$$

Now consider a quaternion operation on a vector **v** such that qvq^{-1} results in a new vector **v**′. That is,

$$\mathbf{v}' = q\mathbf{v}q^{-1} \tag{19-21}$$

Now, consider a Rotation Matrix C that performs a similar operation on vector **v** to yield **v**′ such that

$$\mathbf{v}' = C\mathbf{v} \tag{19-22}$$

or

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{19-23}$$

It is shown in Equation (17-12) of Section 17,

$$q_0 = \frac{1}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}} \tag{19-24}$$

From Equation (19-3) and Equation (19-24):

$$\cos\frac{1}{2}\,\phi = \frac{1}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}} \tag{19-25}$$

Squaring Equation (19-25):

$$\cos^2\frac{1}{2}\,\phi = \frac{1}{4}\left(1 + C_{11} + C_{22} + C_{33}\right) \tag{19-26}$$

Using trigonometry, identify from a Math Table:

$$\cos^2\frac{\phi}{2} = \frac{1 + \cos\phi}{2} \tag{19-27}$$

and substituting Equation (19-27) into Equation (19-26):

$$\frac{1 + \cos\phi}{2} = \frac{1}{4}\left(1 + C_{11} + C_{22} + C_{33}\right) \tag{19-28}$$

It follows:

$$C_{11} + C_{22} + C_{33} = 1 + 2\cos\phi \tag{19-29}$$

Thus, the sum of the diagonal elements, called "trace," of the Rotation Matrix C is equal to $(1 + 2\cos\phi)$. From Equation (19-29):

$$\phi = \cos^{-1}\left[\frac{1}{2}\left(C_{11} + C_{22} + C_{33} - 1\right)\right] \tag{19-30}$$

in which ϕ is the rotation angle about the single equivalent rotation axis, which moves Frame A (primary–reference frame) to Frame B (current Eye orientation).

To confirm that ϕ given in Equation (19-29) is indeed the rotation angle about the Euler axis, consider a rotation by θ_Z about Z–axis. Then, the Rotation Matrix from Frame A to Frame B is, as shown in Section 5 by:

$$C_A^B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta_Z & \sin\theta_Z & 0 \\ -\sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19-31}$$

Applying Equation (19-29):

$$\begin{aligned} C_{11} + C_{22} + C_{33} &= \cos\theta_Z + \cos\theta_Z + 1 \\ &= 1 + 2\cos\theta_Z \end{aligned} \tag{19-32}$$

Comparing Equation (19-29) with Equation (19-32):

$$1 + 2\cos\phi = 1 + 2\cos\theta_Z \tag{19-33}$$

or

$$\phi = \theta_Z \tag{19-34}$$

as expected.

Similarly, for the rotation about X–axis by θ_X:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & \sin\theta_X \\ 0 & -\sin\theta_X & \cos\theta_X \end{bmatrix} \tag{19-35}$$

$$\begin{aligned} C_{11} + C_{22} + C_{33} &= 1 + \cos\theta_X + \cos\theta_X \\ &= 1 + 2\cos\theta_X \end{aligned} \tag{19-36}$$

Comparing Equation (19-29) with Equation (19-36):

$$1 + 2\cos\phi = 1 + 2\cos\theta \tag{19-37}$$

or

$$\phi = \theta_X \tag{19-38}$$

as expected.

SECTION 20

RODRIGUES VECTOR

According to Euler's Theorem, any sequence of rotations of a rigid body (such as an eye–ball or the eye–coil frame fixed on it), that has one point fixed at the origin of the coordinate frame can be achieved by a single equivalent rotation through some angle φ (called Euler's principal angle) about some axis (called Euler's axis) that passes through the same fixed point at the origin of the coordinate frame.

Referring to Section 2, suppose the unit vector **u** along Euler's axis makes an angle α with the X_H axis of Head frame and the X_E axis of Eye frame; an angle β with the Y_H axis and Y_E axis; and an angle γ with the Z_H axis and Z_E axis.

Then the components u_X, u_Y, and u_Z of the unit vector along the X–axis, Y–axis and Z–axis of both frames are

$$u_X = \cos\alpha\,; \quad u_Y = \cos\beta\,; \quad u_Z = \cos\gamma \tag{20-1}$$

and

$$\mathbf{u} = \mathbf{i}u_X + \mathbf{j}u_Y + \mathbf{k}u_Z \tag{20-2}$$

In Section 19, we defined the parameters for the quaternion q by,

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \tag{20-3}$$

by

$$q_0 = \cos\frac{\phi}{2}$$

$$q_1 = \cos\alpha\sin\frac{\phi}{2} = u_x\sin\frac{\phi}{2}$$

$$q_2 = \cos\beta\sin\frac{\phi}{2} = u_y\sin\frac{\phi}{2}$$

$$q_3 = \cos\gamma\sin\frac{\phi}{2} = u_z\sin\frac{\phi}{2} \tag{20-4}$$

where ϕ is Euler's principal angle for the single equivalent rotation.

Note that Equation (20-4) is defined in terms of $\frac{\phi}{2}$, not ϕ.

Rewrite Equation (20-3) as

$$q = s(q) + \mathbf{v}(q) \tag{20-5}$$

where

$$s(q) = (\text{scalor part of } q) = q_0$$

$$\mathbf{v}(q) = (\text{vector part of } q) = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

Using the definition of Equation (20-5), we express the vector part of q as:

$$\begin{aligned} \mathbf{v}(q) &= \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \\ &= \mathbf{i}u_x \ \sin\frac{\phi}{2} + \mathbf{j}u_y \ \sin\frac{\phi}{2} + \mathbf{k}u_z \ \sin\frac{\phi}{2} \\ &= (\mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z)\sin\frac{\phi}{2} \\ &= \mathbf{u} \ \sin\frac{\phi}{2} \end{aligned} \tag{20-6}$$

Equation (20-6) shows that the vector part ($\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$) is equal to the unit vector $\mathbf{u}$ along Euler's axis scaled by $\sin\frac{\phi}{2}$. We modify $\mathbf{v}(q)$ by dividing it by $\cos\frac{\phi}{2} = q_0$ and call it the **Rodrigues Vector** $\boldsymbol{\rho}$ (or some call it Gibbs Vector). The Rodrigues Vector $\boldsymbol{\rho}$ is defined by:

$$\boldsymbol{\rho} \triangleq \frac{\mathbf{u} \ \sin\frac{\phi}{2}}{\cos\frac{\phi}{2}} = \mathbf{u} \ \tan\frac{\phi}{2} \tag{20-7}$$

where $\triangleq$ implies "is equal to" by definition.

From Equation (20-7) with Equation (20-2):

$$\boldsymbol{\rho} = \frac{\mathbf{i}u_x \sin\frac{\phi}{2} + \mathbf{j}u_y \sin\frac{\phi}{2} + \mathbf{k}u_z \sin\frac{\phi}{2}}{\cos\frac{\phi}{2}} \tag{20-8}$$

It follows using Equation (20-4):

$$\boldsymbol{\rho} = \mathbf{i}\frac{q_1}{q_0} + \mathbf{j}\frac{q_2}{q_0} + \mathbf{k}\frac{q_3}{q_0} \tag{20-9}$$

or

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} \dfrac{q_1}{q_0} \\ \dfrac{q_2}{q_0} \\ \dfrac{q_3}{q_0} \end{bmatrix} \tag{20-10}$$

Note, in Equation (20-7), $\boldsymbol{\rho}$ goes to infinity when the denominator $\cos\frac{\phi}{2} = 0$ or $\frac{\phi}{2} = 90^0$ or $\phi = 180^0$. This singularity at $\phi = 180^0$ of $\boldsymbol{\rho}$ is a limitation in the use of the Rodrigues vector, while referring to Equation (20-4), the magnitude of the quaternion parameters q_0, q_1, q_2 and q_3 cannot exceed unity. An advantage of the Rodrigues parameters over the quaternion parameter is that the former has three parameters while the latter has four parameters.

However, the singularity at $\phi = 180^0$ poses no problem for eye movement analysis because this situation equivalent to the rotation of the primary, reference eye position 180^0 to the back of the head, is physiologically impossible movement.

In Section 17, we found that:

$$q_0 = \frac{1}{2}\ (1 + C_{11} + C_{22} + C_{33})^{1/2} \tag{20-11}$$

$$q_1 = \frac{C_{23} - C_{32}}{2\ (1 + C_{11} + C_{22} + C_{33})^{1/2}} \tag{20-12}$$

$$q_2 = \frac{C_{31} - C_{13}}{2\ (1 + C_{11} + C_{22} + C_{33})^{1/2}} \tag{20-13}$$

$$q_3 = \frac{C_{12} - C_{21}}{2\ (1 + C_{11} + C_{22} + C_{33})^{1/2}} \tag{20-14}$$

Using the above equations, we can express the Rodrigues parameters in terms of C_{ij} of the Rotation Matrix C.

It follows from Equation (20-10), using Equation (20-11) to Equation (20-14):

$$\rho_x = \frac{q_1}{q_0} = \frac{C_{23} - C_{32}}{1 + C_{11} + C_{22} + C_{33}} = \frac{R_{32} - R_{23}}{1 + R_{11} + R_{22} + R_{33}} \tag{20-15}$$

$$\rho_y = \frac{q_2}{q_0} = \frac{C_{31} - C_{13}}{1 + C_{11} + C_{22} + C_{33}} = \frac{R_{13} - R_{31}}{1 + R_{11} + R_{22} + R_{33}} \tag{20-16}$$

$$\rho_z = \frac{q_3}{q_0} = \frac{C_{12} - C_{21}}{1 + C_{11} + C_{22} + C_{33}} = \frac{R_{21} - R_{12}}{1 + R_{11} + R_{22} + R_{33}} \tag{20-17}$$

eliminating the square roots present in the denominators of Equation (20-11) to Equation (20-14). For the meanings of R_{ij} vs C_{ij}, see Section 5.

In Section 17, we found the relationship between the elements C_{ij} of the Rotation Matrix C and the parameters q_0, q_1, q_2, and q_3 of quaternion q:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_0 q_3 + q_1 q_2) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_0 q_1 + q_2 q_3) \\ 2(q_0 q_2 + q_1 q_3) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{20-18}$$

From Equation (20-10), we have

$$q_1 = \rho_1 q_0 \;;\; q_2 = \rho_2 q_0 \;;\; q_3 = \rho_3 q_0 \tag{20-19}$$

$$\left(\rho_1 = \rho_X,\ \rho_2 = \rho_Y,\ \rho_3 = \rho_Z\right)$$

Previously in Section 19, Equation (19-10), we found that

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{20-20}$$

Substituting Equation (20-19) into Equation (20-20):

$$q_0^2 + \rho_1^2 q_0^2 + \rho_2^2\ \rho_0^2 + \rho_3^2\ q_0^2 = q_0^2\ (1 + \rho_1^2 + \rho_2^2 + \rho_3^2) = 1 \tag{20-21}$$

or

$$q_0^2 = \frac{1}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2} \tag{20-22}$$

Using Equation (20-19) and Equation (20-22) in several elements of Equation (20-18):

$$C_{11} = q_0^2 + q_1^2 - q_2^2 - q_3^2$$

$$= q_0^2 + \rho_1^2 q_0^2 - \rho_2^2 q_0^2 - \rho_3 q_0^2$$

$$= q_0^2\ (1 + \rho_1^2 - \rho_2^2 - \rho_3^2) \qquad (20\text{-}23)$$

$$C_{11} = \frac{1}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2}\ (1 + \rho_1^2 - \rho_2^2 - \rho_3^2) \qquad (20\text{-}24)$$

$$C_{12} = 2\ (q_0 q_3 + q_1 q_2)$$

$$= 2\ (q_0 \rho_3 q_0 + \rho_1 q_0 \rho_2 q_0)$$

$$= 2\ q_0^2\ (\rho_3 + \rho_1 \rho_2)$$

$$C_{12} = \frac{2}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2}\ (\rho_3 + \rho_1 \rho_2) \qquad (20\text{-}25)$$

Similarly, the rest of C_{ij} may be found in terms of ρ_1, ρ_2 and ρ_3. The results are:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \frac{1}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2}$$

$$\begin{bmatrix} 1 + \rho_1^2 - \rho_2 - \rho_3^2 & 2(\rho_3 + \rho_1 \rho_2) & 2(\rho_1 \rho_3 - \rho_2) \\ 2(\rho_1 \rho_2 - \rho_3) & 1 - \rho_1^2 + \rho_2^2 - \rho_3^2 & 2(\rho_1 + \rho_2 \rho_3) \\ 2(\rho_2 + \rho_1 \rho_3) & 2(\rho_2 \rho_3 - \rho_1) & 1 - \rho_1^2 - \rho_2^2 + \rho_3^2 \end{bmatrix} \qquad (20\text{-}26)$$

To determine Euler principal angle ϕ for single equivalent rotation, we use the previously found relation:

$$C_{11} + C_{22} + C_{33} = 1 + 2\cos\phi \qquad (20\text{-}27)$$

in which C_{11}, C_{22}, and C_{33} may be computed in terms of ρ_1, ρ_2 and ρ_3 from Equation (20-26).

SECTION 21

RODRIGUES VECTOR DIFFERENTIAL EQUATION

In this section, we derive the differential equation for the Rodrigues Vector driven by the angular velocity of the Eye frame relative to the head acting as input.

Referring to Section 20, the Rodrigues Vector $\boldsymbol{\rho}$ is defined by:

$$\boldsymbol{\rho} = \mathbf{u} \tan\frac{\phi}{2} \tag{21-1}$$

where $\mathbf{u}$ is the unit Euler's Rotation vector and ϕ is the (single equivalent) rotation angle about Euler's Rotation axis. It follows:

$$\begin{aligned} \frac{d\boldsymbol{\rho}}{dt} &= \frac{d}{dt}\left(\mathbf{u}\tan\frac{\phi}{2}\right) \\ &= \tan\frac{\phi}{2}\frac{d\mathbf{u}}{dt} + \mathbf{u}\frac{d}{dt}\tan\frac{\phi}{2} \\ &= \tan\frac{\phi}{2}\frac{d\mathbf{u}}{dt} + \frac{\mathbf{u}}{2}\sec^2\frac{\phi}{2}\frac{d\phi}{dt} \end{aligned} \tag{21-2}$$

using the identity $\frac{d}{dt}\tan A = \sec^2 A\frac{dA}{dt}$ where A is an angle.

In Sections 18 and 22, respectively, we derive:

$$\frac{d\mathbf{u}}{dt} = \left[u^* - \left(\cot\frac{\phi}{2}\right)\left(\mathbf{u}\mathbf{u}^T - I\right)\right]\frac{\mathbf{w}}{2} \tag{21-3}$$

$$\frac{d\phi}{dt} = w^T\mathbf{u} \tag{21-4}$$

Substituting Equation (21-3) and Equation (21-4) into Equation (21-2):

$$\begin{aligned} \frac{d\boldsymbol{\rho}}{dt} = {} & \tan\frac{\phi}{2}\left[u^* - \left(\cot\frac{\phi}{2}\right)\left(\mathbf{u}\mathbf{u}^T - I\right]\frac{\mathbf{w}}{2}\right. \\ & + \frac{1}{2}\left(\sec^2\frac{\phi}{2}\right)\mathbf{u}\left(\mathbf{w}^T\mathbf{u}\right) \end{aligned} \tag{21-5}$$

Referring to the last term in Equation (21-5),

$$\mathbf{w}^T\mathbf{u} = \mathbf{u}^T\mathbf{w} = w_X\,\mathbf{u}_X + w_Y\,\mathbf{u}_Y + w_Z\,\mathbf{u}_Z = \mathbf{w}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{w}$$

which is a scalar.

Thus, we can write:

$$\mathbf{u}\mathbf{w}^T\mathbf{u} = \mathbf{u}\left(\mathbf{u}^T\mathbf{w}\right) = \left(\mathbf{u}\mathbf{u}^T\right)\mathbf{w} \tag{21-6}$$

Substituting Equation (21-6) into Equation (21-5) for $\mathbf{u}\left(\mathbf{w}^T\mathbf{u}\right)$, and factoring out $\mathbf{w}$:

$$\begin{aligned}
\frac{d\boldsymbol{\rho}}{dt} &= \left\{\tan\frac{\phi}{2}\left[u^* - \left(\cot\frac{\phi}{2}\right)\left(\mathbf{u}\mathbf{u}^T - I\right)\right] + \left(\sec^2\frac{\phi}{2}\right)\mathbf{u}\mathbf{u}^T\right\}\frac{\mathbf{w}}{2} \\
&= \left\{\left[\tan\frac{\phi}{2}u^* - \left(\tan\frac{\phi}{2}\right)\left(\cot\frac{\phi}{2}\right)\left(\mathbf{u}\mathbf{u}^T - I\right)\right] + \left(\sec^2\frac{\phi}{2}\right)\mathbf{u}\mathbf{u}^T\right\}\frac{\mathbf{w}}{2} \\
&= \left[\left(\tan\frac{\phi}{2}\right)u^* - \mathbf{u}\mathbf{u}^T + I + \left(\sec^2\frac{\phi}{2}\right)\mathbf{u}\mathbf{u}^T\right]\frac{\mathbf{w}}{2} \\
&= \left[\left(\tan\frac{\phi}{2}\right)u^* + \left(\sec^2\frac{\phi}{2} - 1\right)\mathbf{u}\mathbf{u}^T + I\right]\frac{\mathbf{w}}{2} \\
&= \left[\left(\tan\frac{\phi}{2}\right)u^* + \left(\tan^2\frac{\phi}{2}\right)\mathbf{u}\mathbf{u}^T + I\right]\frac{\mathbf{w}}{2}
\end{aligned} \tag{21-7}$$

where trigonometry identities

$$\tan\frac{\phi}{2}\cot\frac{\phi}{2} = 1 \quad \text{and}$$

$$\tan^2\frac{\phi}{2} = \sec^2\frac{\phi}{2} - 1 \quad \text{are used.}$$

By definition of the Rodrigues Vector $\boldsymbol{\rho}$,

$$\boldsymbol{\rho} = \left(\tan\frac{\phi}{2}\right)\mathbf{u}$$

$$\rho^* = \tan\frac{\phi}{2}u^* \tag{21-8}$$

which is, in expanded form (of skew–symmetric matrix):

$$\rho^* = \begin{bmatrix} 0 & -\rho_Z & \rho_Y \\ \rho_Z & 0 & -\rho_X \\ -\rho_Y & \rho_X & 0 \end{bmatrix} = \tan\frac{\phi}{2}\begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} = \tan\frac{\phi}{2}u^* .$$

It follows from Equation (21-8):

$$\boldsymbol{\rho}^T\boldsymbol{\rho} = \left(\tan\frac{\phi}{2}\mathbf{u}^T\right)\left(\tan\frac{\phi}{2}\mathbf{u}\right)$$

$$= \tan^2\frac{\phi}{2}\mathbf{u}^T\mathbf{u}. \tag{21-9}$$

Using Equation (21-8) and Equation (21-9) in Equation (21-7), we have

$$\frac{d\boldsymbol{\rho}}{dt} = \left[I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]\frac{\mathbf{w}}{2}. \tag{21-10}$$

Equation (21-10) may also be expressed in vector form

$$\frac{d\boldsymbol{\rho}}{dt} = \frac{1}{2}[\mathbf{w} + \boldsymbol{\rho}\mathrm{x}\mathbf{w} + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \mathbf{w})] \tag{21-11}$$

in which we used $\left(\boldsymbol{\rho}\boldsymbol{\rho}^T\right)\mathbf{w} = \boldsymbol{\rho}\left(\boldsymbol{\rho}^T\mathbf{w}\right) = \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \mathbf{w})$, and $\rho^*\mathbf{w}$ is replaced by the vector cross–product $\boldsymbol{\rho} \times \mathbf{w}$.

Equation (21-10) or Equation (21-11) is the differential equation of the Rodrigues vector, driven by the angular velocity of the Eye frame relative to the Head frame with components expressed in Eye frame, that is, driven by $\mathbf{w}_{HE}^E$.

Pre–multiplying both sides of Equation (21-10) from the left by the inverse of $\left[I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]$

we have,

$$\mathbf{w} = 2\left[I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]^{-1} \frac{d\boldsymbol{\rho}}{dt} \tag{21-12}$$

In Section 26, we derive the following equation for **w**:

$$\mathbf{w} = \frac{2}{1 + \boldsymbol{\rho}^T\boldsymbol{\rho}} \left[I - \rho^*\right] \frac{d\boldsymbol{\rho}}{dt} \tag{21-13}$$

For Equation (21-12) to be consistent with Equation (21-13), we have to show that

$$\left[I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]^{-1} = \frac{\left[I - \rho^*\right]}{1 + \boldsymbol{\rho}^T\boldsymbol{\rho}} \tag{21-14}$$

To validate the above expression we must show that:

$$\frac{\left[I - \rho^*\right]}{1 + \boldsymbol{\rho}^T\boldsymbol{\rho}} \left[I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right] = I \tag{21-15}$$

(which is to be proven).

Referring to the left side of Equation (21-15):

$$\left[I - \rho^*\right]\left[I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]$$

$$= I + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T - \rho^* - \rho^*\rho^* - \rho^*\boldsymbol{\rho}\boldsymbol{\rho}^T$$

$$= I + \boldsymbol{\rho}\boldsymbol{\rho}^T - \rho^*\rho^* - \rho^*\boldsymbol{\rho}\boldsymbol{\rho}^T \tag{21-16}$$

Now

$$\rho^{*}\boldsymbol{\rho} = \begin{bmatrix} 0 & -\rho_Z & \rho_Y \\ \rho_Z & 0 & -\rho_X \\ -\rho_Y & \rho_X & 0 \end{bmatrix} \begin{bmatrix} \rho_X \\ \rho_Y \\ \rho_Z \end{bmatrix}$$

$$= \begin{bmatrix} -\rho_Z\,\rho_Y + \rho_Y\,\rho_Z \\ \rho_Z\,\rho_X - \rho_X\,\rho_Z \\ -\rho_Y\,\rho_X + \rho_X\,\rho_Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (21\text{-}17)$$

$$\rho^{*}\rho^{*} = \begin{bmatrix} 0 & -\rho_Z & \rho_Y \\ \rho_Z & 0 & -\rho_X \\ -\rho_Y & \rho_X & 0 \end{bmatrix} \begin{bmatrix} 0 & -\rho_Z & \rho_Y \\ \rho_Z & 0 & -\rho_X \\ -\rho_Y & \rho_X & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\rho_Z^2 - \rho_Y^2 & \rho_Y\ \rho_X & \rho_Z\ \rho_X \\ \rho_X\ \rho_Y & -\rho_Z^2 - \rho_Y^2 & \rho_Z\ \rho_Y \\ \rho_X\ \rho_Z & \rho_Y\ \rho_Z & -\rho_Y^2 - \rho_X^2 \end{bmatrix} \qquad (21\text{-}18)$$

$$\boldsymbol{\rho\rho}^T - \boldsymbol{\rho}^T\boldsymbol{\rho}\mathbf{I} = \left(\begin{bmatrix} \rho_X \\ \rho_Y \\ \rho_Z \end{bmatrix} \begin{bmatrix} \rho_X \rho_Y \rho_Z \end{bmatrix} \right) - \left(\begin{bmatrix} \rho_X \rho_Y \rho_Z \end{bmatrix} \begin{bmatrix} \rho_X \\ \rho_Y \\ \rho_Z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \rho_X^2 & \rho_X\,\rho_Y & \rho_X\,\rho_Z \\ \rho_Y\,\rho_X & \rho_Y^2 & \rho_Y\,\rho_Z \\ \rho_Z\,\rho_X & \rho_Z\,\rho_Y & \rho_Z^2 \end{bmatrix}$$

$$- \begin{bmatrix} \rho_X^2 + \rho_Y^2 + \rho_Z^2 & 0 & 0 \\ 0 & \rho_X^2 + \rho_Y^2 + \rho_Z^2 & 0 \\ 0 & 0 & \rho_X^2 + \rho_Y^2 + \rho_Z^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\rho_Y^2 - \rho_Z^2 & \rho_X\,\rho_Y & \rho_X\,\rho_Z \\ \rho_Y\,\rho_X & -\rho_X^2 - \rho_Y^2 & \rho_Y\,\rho_Z \\ \rho_Z\,\rho_X & \rho_Z\,\rho_Y & -\rho_X^2 - \rho_Y^2 \end{bmatrix} \qquad (21\text{-}19)$$

Comparing Equation (21-18) and Equation (21-19), we conclude:

$$\rho^*\rho^* = \boldsymbol{\rho}\boldsymbol{\rho}^T - \boldsymbol{\rho}^T\boldsymbol{\rho}\mathrm{I} \tag{21-20}$$

Substituting Equation (21-17) and Equation (21-20) into Equation (21-16):

$$\begin{aligned}
&\left[\mathrm{I} - \rho^*\right]\ \left[\mathrm{I} + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right] \\
&\quad = \mathrm{I} + \boldsymbol{\rho}\boldsymbol{\rho}^T - \rho^*\rho^* - \left(\rho^*\rho\right)\boldsymbol{\rho}^T \\
&\quad = \mathrm{I} + \boldsymbol{\rho}\boldsymbol{\rho}^T - \boldsymbol{\rho}\boldsymbol{\rho}^T + \boldsymbol{\rho}^T\boldsymbol{\rho}\mathrm{I} - 0 \\
&\quad = \mathrm{I} + \boldsymbol{\rho}^T\boldsymbol{\rho}\mathrm{I} \\
&\quad = \left(1 + \boldsymbol{\rho}^T\boldsymbol{\rho}\right)\mathrm{I}
\end{aligned} \tag{21-21}$$

Substituting Equation (21-21) into Equation (21-15), we have

$$\frac{\left[\mathrm{I} - \rho^*\right]\left[\mathrm{I} + \rho^* + \boldsymbol{\rho}\boldsymbol{\rho}^T\right]}{1 + \boldsymbol{\rho}^T\boldsymbol{\rho}} = \frac{(1 + \boldsymbol{\rho}^T\boldsymbol{\rho})\mathrm{I}}{1 + \boldsymbol{\rho}^T\boldsymbol{\rho}} = \mathrm{I} \tag{21-22}$$

QED.

SECTION 22

ROTATION ANGULAR RATE ABOUT THE EULER AXIS

In Equation (17-30), we found that

$$C_{11} + C_{22} + C_{33} = 1 + 2\cos\phi \tag{22-1}$$

where ϕ is the rotation angle about the Euler Axis of the Eye frame relative to the Head frame, and C_{11}, C_{22}, and C_{33} are the diagonal elements of the rotation matrix from Frame E to Frame H, as shown below:

$$\mathbf{r}^H = C_E^H\,\mathbf{r}^E \tag{22-2}$$

$$\begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} \tag{22-3}$$

Note that the trace of C_E^H is the same as the trace of C_H^E, being the sum of the diagonal elements that are common to both matrices.

From Equation (22-1):

$$\cos\phi = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \tag{22-4}$$

By definition, $C_{11} + C_{22} + C_{33}$ is called the trace of C, denoted by tr C. Using this notation in Equation (22-4):

$$\cos\phi = \frac{1}{2}\left[\mathrm{tr}\ C_H^E - 1\right] \tag{22-5}$$

We want to find $\frac{d\phi}{dt}$.

Differentiating both sides of Equation (22-5) with respect to time:

$$\sin\phi\frac{d\phi}{dt} = \frac{1}{2}\left[\frac{d}{dt}\mathrm{tr}\ C_H^E - 0\right]$$

$$= \frac{1}{2}\mathrm{tr}\left[\frac{d}{dt}C_H^E\right] \tag{22-6}$$

In Section 15, we found:

$$\frac{d}{dt} C_E^H = C_E^H \, w_{HE}^{E*} \tag{22-7}$$

where w_{HE}^{E*} is the matrix form of the angular velocity $\mathbf{w}_{HE}^{E}$ of Frame E relative to Frame H, with the superscript E indicating that its components are expressed in the Frame E.

Taking the transpose of both sides of Equation (22-7):

$$\frac{d}{dt}\left[C_E^H\right]^T = \left[C_E^H \, w_{HE}^{E*}\right]^T$$

$$= \left[w_{HE}^{E*}\right]^T \left[C_E^H\right]^T \tag{22-8}$$

Now:

$$\left[C_E^H\right]^T = C_H^E \tag{22-9}$$

$$\left[w_{HE}^{E*}\right]^T = \begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & w_Z & -w_Y \\ -w_Z & 0 & w_X \\ w_Y & -w_X & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}$$

$$= -\, w_{HE}^{E*} \tag{22-10}$$

Substituting Equation (22-9) and Equation (22-10) into Equation (22-8):

$$\frac{d}{dt}\left[C_H^E\right] = -\, w_{HE}^{E*} \, C_H^E \tag{22-11}$$

Substituting Equation (22-11) into Equation (22-6):

$$\sin\phi \frac{d\phi}{dt} = -\frac{1}{2} \, \mathrm{tr}\left[w_{HE}^{E*} \, C_H^E\right] \tag{22-12}$$

It follows:

$$\frac{d\phi}{dt} = -\frac{1}{2\sin\phi}\ \mathrm{tr}\left[w_{HE}^{E*}\ C_H^E\right] \tag{22-13}$$

Recalling that $C_H^E = \left[C_E^H\right]^T$, and referring to Equation (22-3):

$$\frac{d\phi}{dt} = -\frac{1}{2\sin\phi}\ \mathrm{tr}\left\{\begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\right\} \tag{22-14}$$

Remembering that the trace of a matrix is the sum of the diagonal elements or (1,1), (2,2) and (3,3) elements, we have after expanding and summing diagonal elements:

$$\frac{d\phi}{dt} = -\frac{1}{2\sin\phi}\ \left[(-w_Z\, C_{12} + w_Y\, C_{13}) + (w_Z\, C_{21} - w_x\, C_{23}) + (-w_Y\, C_{31} + w_x\, C_{32})\right]$$

$$= +\frac{1}{2\sin\phi}\ \left[w_X\,(C_{23} - C_{32}) + w_Y\,(C_{31} - C_{13}) + w_Z\,(C_{12} - C_{21})\right] \tag{22-15}$$

In vector form, the matrix $\left[w_{HE}^{E*}\right]$ becomes

$$\mathbf{w}_{HE} = \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix} \quad \text{and} \quad \mathbf{w}_{HE}^T = \left[w_X\ w_Y\ w_Z\right] \tag{22-16}$$

In Section 17, the unit vector along the Euler axis was found to be:

$$\mathbf{u} = \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \frac{1}{2\sin\phi}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix} \tag{22-17}$$

where C_{ij} in Equation (22-17) are the elements of C_E^H, which is equal to the transpose of C_H^E.

Using Equation (22-16) and Equation (22-17):

$$\mathbf{w}_{HE}^T\ \mathbf{u} = \frac{1}{2\sin\phi}\ \left[w_X\left(C_{23} - C_{32}\right) + w_Y\left(C_{31} - C_{13}\right) + w_Z\left(C_{12} - C_{21}\right)\right] \tag{22-18}$$

Comparing Equation (22-15) and Equation (22-18), we conclude:

$$\frac{d\phi}{dt} = \left(\mathbf{w}_{HE}^{E}\right)^{T} \quad \mathbf{u} \tag{22-19}$$

Note that $\left(\mathbf{w}_{HE}^{E}\right)^{T} = \left[w_X w_Y w_Z\right]$ is in a vector form, while

$$\left[w_{HE}^{E*}\right] = \begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}$$

is in matrix form of the angular velocity of the Eye frame relative to the Head frame.

Equation (22-19) is a compact form of Equation (22-15), which expresses the Euler angle rate $\frac{d\phi}{dt}$. Equation (22-15) expresses $\frac{d\phi}{dt}$ in terms of ϕ, and components of w and C, while Equation (22-19) expresses $\frac{d\phi}{dt}$ in terms of vectors **w** and **u**.

SECTION 23

UNIFIED FORM OF EULER'S ROTATION VECTOR RATE AND ROTATION ANGULAR RATE ABOUT THE EULER AXIS

In Section 18, we derived the following equation for the Euler's Rotation Vector Rate:

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \frac{1}{2}\left[u^* - \cos\left(\frac{\phi}{2}\right)\left(\mathbf{u}\mathbf{u}^T - I\right)\right]\mathbf{w}_{HE}^{E} \tag{23-1}$$

where

$\mathbf{u}$ = unit vector along the Euler's Rotation axis.

$$u^* = \begin{bmatrix} 0 & -u_Z & u_Y \\ u_Z & 0 & -u_X \\ -u_Y & u_X & 0 \end{bmatrix} \tag{23-2}$$

= a Skew–symmetric Matrix corresponding to vector $\mathbf{u} = \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix}$.

ϕ = scalar rotation angle about the Euler's Rotation Axis.

$\mathbf{w}_{HE}^{E}$ = angular velocity vector of the Eye frame (Rotating frame) relative to the Head frame (Reference frame) with the components expressed in the superscript Frame E.

Also in Section 22, we derived the equation for the Rotation Angular Rate about the Euler Axis given below:

$$\dot{\phi} = \frac{d\phi}{dt} = \left(\mathbf{w}_{HE}^{E}\right)^T \mathbf{u} \tag{23-3}$$

in which ϕ and $\dot{\phi}$ are scalar variables.

To achieve the unified form, combining the equations for $\dot{\mathbf{u}}$ and $\dot{\phi}$, we define a new vector variable $\boldsymbol{\phi}$ by

$$\boldsymbol{\phi} = \phi\mathbf{u} \tag{23-4}$$

in which $\boldsymbol{\phi}$ has the same direction as $\mathbf{u}$ with its magnitude equal to ϕ since $\mathbf{u}$ is a unit vector. That is, $|\boldsymbol{\phi}| = \phi$.

It follows from Equation (23-4):

$$\mathbf{u} = \frac{\boldsymbol{\phi}}{\phi} \tag{23-5}$$

Differentiating Equation (23-5):

$$\frac{d\mathbf{u}}{dt} = \frac{1}{\phi}\dot{\boldsymbol{\phi}} - \frac{1}{\phi^2}\dot{\phi}\,\boldsymbol{\phi} \tag{23-6}$$

Solving Equation (23-6) for $\dot{\boldsymbol{\phi}}$ and using Equation (23-5):

$$\dot{\boldsymbol{\phi}} = \phi\frac{d\mathbf{u}}{dt} + \frac{1}{\phi}\dot{\phi}\,\boldsymbol{\phi}$$

$$= \phi\frac{d\mathbf{u}}{dt} + \mathbf{u}\,\dot{\phi} \tag{23-7}$$

By direct substitutions for u^* from Equation (23-2), it is easy to show that

$$\mathbf{u}\mathbf{u}^T - I = u^*u^* \tag{23-8}$$

or

$$\mathbf{u}\mathbf{u}^T = I + u^*u^* \tag{23-9}$$

where $\mathbf{u}^T = \begin{bmatrix} u_X & u_Y & u_Z \end{bmatrix}$.

Now, substituting Equation (23-1) for $\frac{d\mathbf{u}}{dt}$ and Equation (23-3) for $\dot{\phi}$ in Equation (23-7), and denoting $\mathbf{w}^E_{HE}$ by $\mathbf{w}$ for simplicity:

$$\dot{\boldsymbol{\phi}} = \phi\left[\frac{1}{2}u^* - \frac{1}{2}\cot\left(\frac{\phi}{2}\right)\left(\mathbf{u}\mathbf{u}^T - I\right)\right]\mathbf{w} + \mathbf{u}\mathbf{u}^T\mathbf{w} \tag{23-10}$$

where we used $\mathbf{w}^T\mathbf{u} = \mathbf{u}^T\mathbf{w}$ corresponding to the dot product $\mathbf{w} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{w}$.

From Equation (23-5):

$$\mathbf{u} = \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \begin{bmatrix} \frac{\phi_X}{\phi} \\ \frac{\phi_Y}{\phi} \\ \frac{\phi_Z}{\phi} \end{bmatrix} = \frac{1}{\phi}\begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \frac{\boldsymbol{\phi}}{\phi} \tag{23-11}$$

Substituting Equation (23-11) into Equation (23-2):

$$u^* = \begin{bmatrix} 0 & \frac{-\phi_Z}{\phi} & \frac{\phi_Y}{\phi} \\ \frac{\phi_Z}{\phi} & 0 & \frac{-\phi_X}{\phi} \\ \frac{-\phi_Y}{\phi} & \frac{\phi_X}{\phi} & 0 \end{bmatrix}$$

$$= \frac{1}{\phi}\begin{bmatrix} 0 & -\phi_Z & \phi_Y \\ \phi_Z & 0 & -\phi_X \\ -\phi_Y & \phi_X & 0 \end{bmatrix}$$

$$= \frac{\phi^*}{\phi} \tag{23-12}$$

It follows:

$$u^* u^* = \frac{1}{\phi^2}\phi^*\phi^* \tag{23-13}$$

Using Equation (23-9), Equation (23-12) and Equation (23-13) in Equation (23-10):

$$\dot{\boldsymbol{\phi}} = \phi\left[\frac{1}{2}\frac{\boldsymbol{\phi}^*}{\phi} - \frac{1}{2}\cot\left(\frac{\phi}{2}\right)\frac{1}{\phi^2}\,\boldsymbol{\phi}^*\,\boldsymbol{\phi}^*\right]\mathbf{w} + \left(\mathbf{I} + \mathbf{u}^*\,\mathbf{u}^*\right)\mathbf{w}$$

$$= \frac{1}{2}\,\boldsymbol{\phi}^*\,\mathbf{w} - \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right)\frac{1}{\phi^2}\,\boldsymbol{\phi}^*\boldsymbol{\phi}^*\,\mathbf{w} + \mathbf{w} + \frac{1}{\phi^2}\,\boldsymbol{\phi}^*\,\boldsymbol{\phi}^*\,\mathbf{w}$$

$$= \mathbf{w} + \frac{1}{2}\,\boldsymbol{\phi}^*\,\mathbf{w} + \frac{1}{\phi^2}\left[1 - \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right)\right]\boldsymbol{\phi}^*\,\boldsymbol{\phi}^*\,\mathbf{w}$$

$$\dot{\boldsymbol{\phi}} = \mathbf{w} + \frac{1}{2}\boldsymbol{\phi}^*\mathbf{w} + \frac{1}{\phi^2}\left[1 - \frac{\phi}{2}\ \ \frac{\sin\phi}{1-\cos\phi}\right]\boldsymbol{\phi}^*\,\boldsymbol{\phi}^*\,\mathbf{w} \qquad (23\text{-}14)$$

where we used a trigonometry identity:

$$\cot\frac{\phi}{2} = \frac{\sin\phi}{1-\cos\phi} \qquad (23\text{-}15)$$

Equation (23-14) is the unified, combined form of $\dot{\mathbf{u}}$ and $\dot{\phi}$ given in Equation (23-1) and Equation (23-3).

We may change Equation (23-14), which is a matrix differential equation to a vector differential equation, by replacing, in Equation (23-14), $\boldsymbol{\phi}^*\mathbf{w}$ by $\boldsymbol{\phi} \times \mathbf{w}$ and $\boldsymbol{\phi}^*\boldsymbol{\phi}^*\mathbf{w}$ by $\boldsymbol{\phi} \times \boldsymbol{\phi} \times \mathbf{w}$ as shown below:

$$\dot{\boldsymbol{\phi}} = \mathbf{w} + \frac{1}{2}\boldsymbol{\phi} \times \mathbf{w} + \frac{1}{\phi^2}\left(1 - \frac{\phi}{2}\ \ \frac{\sin\phi}{1-\cos\phi}\right)\boldsymbol{\phi} \times \boldsymbol{\phi} \times \mathbf{w}. \qquad (23\text{-}16)$$

The above equation is originally developed by John E. Bortz (see his article listed in the Bibliography 7). The Equation (23-14) is derived here by using entirely different approaches from those originally used in the Bibliography 7 to facilitate the comprehensions by readers, consistent with the developments in this report.

SECTION 24

QUATERNION DIFFERENTIAL EQUATION

Referring to Section 19, we recall a quaternion

$$q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$= q_0 + \mathbf{q} \tag{24-1}$$

where

$$\mathbf{q} = iq_1 + jq_2 + kq_3 = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Equation (24-1) may be expressed by a column vector by: (see Section 19)

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ u_x \sin\frac{\phi}{2} \\ u_y \sin\frac{\phi}{2} \\ u_z \sin\frac{\phi}{2} \end{bmatrix}$$

$$= \begin{bmatrix} q_0 \\ \hdashline \mathbf{q} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\phi}{2} \\ \hdashline \mathbf{u}\sin\frac{\phi}{2} \end{bmatrix} \tag{24-2}$$

where ϕ is the angle of quaternion q, and $\mathbf{u}$ is the unit vector along the Euler Axis of single equivalent rotation and $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$.

From Equation (24-2), we have:

$$\mathbf{q} = \mathbf{u} \sin \frac{\phi}{2} \tag{24-3}$$

Differentiating Equation (24-3) with respect to time:

$$\frac{d\mathbf{q}}{dt} = \frac{1}{2} \cos \frac{\phi}{2} \mathbf{u} \frac{d\phi}{dt} + \sin \frac{\phi}{2} \frac{d\mathbf{u}}{dt} \tag{24-4}$$

From Section 22,

$$\frac{d\phi}{dt} = \left[\mathbf{w}_{HE}^{E}\right]^{T} \mathbf{u} \tag{24-5}$$

And, from Section 18,

$$\frac{d\mathbf{u}}{dt} = \left[\frac{1}{2} u * - \frac{1}{2} \cot \left(\frac{\phi}{2}\right) \left(\mathbf{u}\mathbf{u}^{T} - I\right)\right] \mathbf{w}_{HE}^{E} \tag{24-6}$$

where $\mathbf{w}_{HE}^{E}$ is the angular velocity of the Eye frame relative to the Head frame with the components resolved into the Eye frame.

Substituting Equation (24-5) for $\frac{d\phi}{dt}$ and Equation (24-6) for $\frac{d\mathbf{u}}{dt}$ into Equation (24-4):

$$\frac{d\mathbf{q}}{dt} = \left\{\left[\mathbf{w}_{HE}^{E}\right]^{T} \mathbf{u}\right\} \frac{1}{2} \left(\cos \frac{\phi}{2}\right) \mathbf{u}$$

$$+ \sin \left(\frac{\phi}{2}\right) \left[\frac{1}{2} u^{*} - \frac{1}{2} \cot \left(\frac{\phi}{2}\right) \left(\mathbf{u}\, \mathbf{u}^{T} - I\right)\right] \mathbf{w}_{HE}^{E} \tag{24-7}$$

It follows from Equation (24-7), since

$$\cot \left(\frac{\phi}{2}\right) = \cos \left(\frac{\phi}{2}\right) / \sin \left(\frac{\phi}{2}\right) \quad \text{that:}$$

$$\frac{d\mathbf{q}}{dt} = \frac{1}{2} \sin \frac{\phi}{2} u^{*} \mathbf{w} + \frac{1}{2} \cos \frac{\phi}{2} \left(\mathbf{w}^{T} \mathbf{u}\right) \mathbf{u}$$

$$- \frac{1}{2} \cos \frac{\phi}{2} \mathbf{u}\mathbf{u}^{T} \mathbf{w} + \frac{1}{2} \cos \frac{\phi}{2} \mathbf{w}$$

$$= \frac{1}{2} \sin \frac{\phi}{2} u^{*}\mathbf{w} + \frac{1}{2} \cos \frac{\phi}{2} \left[\left(\mathbf{w}^{T}\mathbf{u}\right) \mathbf{u} - \mathbf{u} \left(\mathbf{u}^{T}\mathbf{w}\right) + \mathbf{w}\right] \tag{24-8}$$

where we used $\mathbf{w}$ for $\mathbf{w}_{HE}^{E}$ for simplicity.

Now $\mathbf{u}^T\mathbf{w} = \mathbf{w}^T\mathbf{u}$ corresponds to a dot product $\mathbf{u} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u}$, which is a scalar (non–vector) quality. We can easily see that

$$\mathbf{u}\mathbf{u}^T\mathbf{w} = \mathbf{u}\left(\mathbf{u}^T\mathbf{w}\right) = \mathbf{u}\left(\mathbf{w}^T\mathbf{u}\right) = \left(\mathbf{w}^T\mathbf{u}\right)\mathbf{u} \tag{24-9}$$

Using Equation (24-9) in Equation (24-8):

$$\frac{d\mathbf{q}}{dt} = \frac{1}{2}\sin\frac{\phi}{2}\,u^*\mathbf{w} + \frac{1}{2}\cos\frac{\phi}{2}\,\mathbf{w} \tag{24-10}$$

We recall the skew–symmetric form w^* corresponding to the vector $\mathbf{w}$ is given by:

$$w^* = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \tag{24-11}$$

Similarly, u^* corresponding to $\mathbf{u}$ is given by:

$$u^* = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \tag{24-12}$$

By direct substitutions of Equation (24-11) and Equation (24-12), it may be easily confirmed that

$$u^*\mathbf{w} = -\,w^*\mathbf{u} \tag{24-13}$$

where

$$\mathbf{w} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

Using Equation (24-13) and Equation (24-2) in Equation (24-10):

$$\frac{d\mathbf{q}}{dt} = -\frac{1}{2}\sin\frac{\phi}{2}\,w^*\mathbf{u} + \frac{1}{2}\,q_0\mathbf{w}$$

$$= -\frac{1}{2}w^*\left(\sin\frac{\phi}{2}\,\mathbf{u}\right) + \frac{1}{2}\,q_0\mathbf{w}$$

$$= -\frac{1}{2}\,w^*\mathbf{q} + \frac{1}{2}\,\mathbf{w}q_0 \tag{24-14}$$

Now

$$\frac{dq_0}{dt} = \frac{d}{dt}\cos\frac{\phi}{2}$$

$$= -\frac{1}{2}\sin\frac{\phi}{2}\,\frac{d\phi}{dt} \tag{24-15}$$

Using Equation (24-5) in Equation (24-15):

$$\frac{dq_0}{dt} = -\frac{1}{2}\sin\frac{\phi}{2}\,\mathbf{w}^T\mathbf{u}$$

$$= -\frac{1}{2}\sin\frac{\phi}{2}\,\mathbf{u}^T\mathbf{w}$$

$$= -\frac{1}{2}\left(\sin\frac{\phi}{2}\,\mathbf{u}\right)^T\mathbf{w} \tag{24-16}$$

It follows from Equation (24-16) using Equation (24-2):

$$\frac{dq_0}{dt} = -\frac{1}{2}\,\mathbf{q}^T\mathbf{w}$$

$$= -\frac{1}{2}\,\mathbf{w}^T\mathbf{q} \tag{24-17}$$

Combining Equation (24-17) and Equation (24-14), and expressing in a matrix form:

$$\frac{dq}{dt} = \begin{bmatrix} \dfrac{dq_0}{dt} \\ \dfrac{d\mathbf{q}}{dt} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -\mathbf{w}^T \\ \mathbf{w} & -w^* \end{bmatrix} \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \tag{24-18}$$

where

$$\mathbf{w} = \mathbf{w}_{HE}^{E} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$$\mathbf{w}^T = \left[\mathbf{w}_{HE}^{E}\right]^T = \begin{bmatrix} w_x & w_y & w_z \end{bmatrix}$$

$$w^* = w_{HE}^{E*} = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Substituting the above equations into Equation (24-18):

$$\frac{d}{dt}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{2}\left[\begin{array}{c|ccc} 0 & -w_x & -w_y & -w_z \\ \hline w_x & 0 & w_z & -w_y \\ w_y & -w_z & 0 & w_x \\ w_z & w_y & -w_x & 0 \end{array}\right] \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{24-19}$$

Expanding Equation (24-19):

$$\frac{d}{dt}q_0 = \frac{1}{2}\left(0q_0 + -w_x q_1 - w_y q_2 - w_z q_3\right)$$

$$\frac{d}{dt}q_1 = \frac{1}{2}\left(w_x q_0 + 0q_1 + w_2 q_2 - w_y q_3\right)$$

$$\frac{d}{dt}q_2 = \frac{1}{2}\left(w_y q_0 - w_2 q_1 + 0q_2 + w_x q_3\right)$$

$$\frac{d}{dt}q_3 = \frac{1}{2}\left(w_z q_0 + \;\; w_y q_1 - w_x q_2 + \;\; 0q_3\right) \tag{24-20}$$

Rearranging Equation (24-20)

$$\frac{d}{dt}q_0 = \frac{1}{2}\left(-\, q_1 w_x - q_2 w_y - q_3 w_z\right)$$

$$\frac{d}{dt}q_1 = \frac{1}{2}\left(q_0 w_x - q_3 w_y + q_2 w_z\right)$$

$$\frac{d}{dt}q_2 = \frac{1}{2}\left(q_3 w_x + q_0 w_y - q_1 w_z\right)$$

$$\frac{d}{dt}q_3 = \frac{1}{2}\left(-\, q_2 w_x + q_1 w_y + q_0 w_z\right) \tag{24-21}$$

Expressing Equation (24-21) in a matrix form:

$$\frac{dq}{dt} = \frac{d}{dt}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \tag{24-22}$$

SECTION 25

ANGULAR VELOCITY OF EYE FRAME RELATIVE TO THE HEAD FRAME IN TERMS OF QUATERNION ELEMENTS

In this section, we derive the angular velocity of the Eye Frame relative to the Head Frame, $\mathbf{w}_{HE}^{E}$, in terms of the quaternion elements q_0, q_1, q_2 and q_3.

Reviewing some of the notations we used in the previous sections,

$$\mathbf{w}_{HE}^{E} = \mathbf{w} = \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix}$$

$$\left[\mathbf{w}_{HE}^{E}\right]^T = \mathbf{w}^T = \left[w_X \; w_Y \; w_Z\right]$$

$$w_{HE}^{E*} = w^* = \begin{bmatrix} 0 & -w_Z & w_Y \\ w_Z & 0 & -w_X \\ -w_Y & w_X & 0 \end{bmatrix}$$

$$\left[w_{HE}^{E*}\right]^T = \begin{bmatrix} 0 & w_Z & -w_Y \\ -w_Z & 0 & w_X \\ w_Y & -w_X & 0 \end{bmatrix}$$

It follows:

$$\left[w_{HE}^{E*}\right]^T = -\, w_{HE}^{E*} \tag{25-1}$$

In Section 24, we derived:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \left[\begin{array}{c|ccc} 0 & -w_X & -w_Y & -w_Z \\ \hline w_X & 0 & w_Z & -w_Y \\ w_Y & -w_Z & 0 & w_X \\ w_Z & w_Y & -w_X & 0 \end{array}\right] \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{25-2}$$

Using notation $\mathbf{q}_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ in which the subscript v refers to the vector part of a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, we may write Equation (25-2) as shown in Equation (24-18):

$$\begin{bmatrix} \dot{q}_0 \\ \dot{\mathbf{q}}_v \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\mathbf{w}^T \\ \mathbf{w} & -w^* \end{bmatrix} \begin{bmatrix} q_0 \\ \mathbf{q}_v \end{bmatrix} \tag{25-3}$$

Expanding Equation (25-3):

$$2\,\dot{q}_0 = -\mathbf{w}^T\mathbf{q}_v \tag{25-4}$$

$$2\,\dot{\mathbf{q}}_v = \mathbf{w}q_0 - w^*\mathbf{q}_v \tag{25-5}$$

Referring to the term $-\,w^*\mathbf{q}_v$ in Equation (25-5):

$$\left[-w^*\right]\mathbf{q}_v = \begin{bmatrix} 0 & w_Z & -w_Y \\ -w_Z & 0 & w_X \\ w_Y & -w_X & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$= \begin{bmatrix} w_Z\,q_2 & -\,w_Y\,q_3 \\ -\,w_Z\,q_1 & +\,w_X\,q_3 \\ w_Y\,q_1 & -\,w_X\,q_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix}$$

$$= q_v^*\mathbf{w} \tag{25-6}$$

where q_v^* is a 3 x 3 matrix equivalent of a 3 x 1 vector $\mathbf{q}_v$. See Section 1. It follows by putting Equation (25-6) into Equation (25-5) for $-w^*\mathbf{q}_v$:

$$2\,\dot{\mathbf{q}}_v = q_v^*\mathbf{w} + \mathbf{w}q_0 \tag{25-7}$$

For any vectors **A** and **B**, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$, or $\mathbf{A}^T\mathbf{B} = \mathbf{B}^T\mathbf{A}$. It follows from Equation (25-4):

$$2\,\dot{q}_0 = -\,\mathbf{w}^T\mathbf{q}_v = -\,\mathbf{q}_v^T\mathbf{w} \tag{25-8}$$

Multiplying both sides of Equation (25-7) by q_v^* from the left:

$$2q_v^* \dot{\mathbf{q}}_v = q_v^* q_v^* \mathbf{w} + q_v^* \mathbf{w} q_0 \tag{25-9}$$

Referring to the term $q_v^* q_v^* \mathbf{w}$ in Equation (25-9), recall the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{C} \cdot \mathbf{A}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, and that $\mathrm{A}^*\mathbf{B}$ is equivalent to the vector cross–product $\mathbf{A} \times \mathbf{B}$, and $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$:

$$\begin{aligned} &= \mathbf{q}_v(\mathbf{w} \cdot \mathbf{q}_v) - \mathbf{w}(\mathbf{q}_v \cdot \mathbf{q}_v) \\ &= \mathbf{q}_v\left(\mathbf{q}_v^T \mathbf{w}\right) - \mathbf{w}\left(\mathbf{q}_v^T \mathbf{q}_v\right) \end{aligned} \tag{25-10}$$

Now:

$$\begin{aligned} \mathbf{q}_v^T \mathbf{q}_v &= \left[q_1 q_2 q_3\right] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \\ &= q_1^2 + q_2^2 + q_3^2 \\ &= \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right) - q_0^2 \\ &= 1 - q_0^2 \end{aligned} \tag{25-11}$$

where $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ (as derived in Section 19) is used.

Using Equation (25-11) in Equation (25-10):

$$\begin{aligned} q_v^* q_v^* \mathbf{w} &= \mathbf{q}_v \mathbf{q}_v^T \mathbf{w} - \mathbf{w}\left(1 - q_0^2\right) \\ &= \mathbf{q}_v \mathbf{q}_v^T \mathbf{w} - \left(1 - q_0^2\right) \mathbf{w} \end{aligned} \tag{25-12}$$

since $\left(1 - q_0^2\right)$ is a scalar.

Using Equation (25-12) in Equation (25-9):

$$2q_v^* \dot{\mathbf{q}}_v = \mathbf{q}_v \mathbf{q}_v^T \mathbf{w} - \left(1 - q_0^2\right) \mathbf{w} + q_v^* \mathbf{w} q_0 \tag{25-13}$$

Now pre–multiplying Equation (25-4) by $\mathbf{q}_v$ on both sides:

$$2\mathbf{q}_v \dot{q}_0 = -\mathbf{q}_v \mathbf{w}^T \mathbf{q}_v = -\mathbf{q}_v \mathbf{q}_v^T \mathbf{w} \tag{25-14}$$

since $\mathbf{w}^T \mathbf{q}_v = \mathbf{q}_v^T \mathbf{w}$ for the dot product.

Putting Equation (25-14) into Equation (25-13) for $\mathbf{q}_v \mathbf{q}_v^T \mathbf{w}$, and transposing:

$$2q_v^* \dot{\mathbf{q}}_v + 2\mathbf{q}_v \dot{q}_0 = -\left(1 - q_0^2\right)\mathbf{w} + q_v^* \mathbf{w} q_0 \tag{25-15}$$

Now multiplying Equation (25-7) by q_0 (scalar):

$$2q_0 \dot{\mathbf{q}}_v = q^* \mathbf{w} q_0 + \mathbf{w} q_0^2 \tag{25-16}$$

It follows:

$$q^* \mathbf{w} q_0 = 2q_0 \dot{\mathbf{q}}_v - \mathbf{w} q_0^2 \tag{25-17}$$

Putting Equation (25-17) into Equation (25-15) for $q^* \mathbf{w} q_0$:

$$\begin{aligned} 2q_v^* \dot{\mathbf{q}}_v + 2\mathbf{q}_v \dot{q}_0 &= -\mathbf{w} + q_0^2 \mathbf{w} + 2q_0 \dot{\mathbf{q}}_v - \mathbf{w} q_0^2 \\ &= -\mathbf{w} + 2q_0 \dot{\mathbf{q}}_v \end{aligned} \tag{25-18}$$

Solving for $\mathbf{w}$:

$$\begin{aligned} \mathbf{w}_{HE}^{E} = \mathbf{w} &= 2\left[q_0 \dot{\mathbf{q}}_v - \dot{q}_0 \mathbf{q}_v - q_v^* \mathbf{q}_v\right] \\ &= 2\left[q_0 \frac{d}{dt}\mathbf{q}_v - \left(\frac{dq_0}{dt}\right)\mathbf{q}_v - q_v^*\left(\frac{d\mathbf{q}_v}{dt}\right)\right] \end{aligned} \tag{25-19}$$

Equation (25-19) is the desired results which expresses the angular velocity of the Eye Frame relative to the Head Frame with components expressed in the Eye frame. That is, $\mathbf{w}_{HE}^{E}$ in terms of the elements of a quaternion, q_0 and $\mathbf{q}_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$ and their derivative with respect to time.

We recall from Section 17 that

$$q_0 = \frac{1}{2}\left(1 + C_{11} + C_{22} + C_{33}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 + R_{11} + R_{22} + R_{33}\right)^{\frac{1}{2}} \tag{25-20}$$

$$q_1 = \frac{C_{23} - C_{32}}{4q_0} = \frac{R_{32} - R_{23}}{4q_0} \tag{25-21}$$

$$q_2 = \frac{C_{31} - C_{13}}{4q_0} = \frac{R_{13} - R_{31}}{4q_0} \tag{25-22}$$

$$q_3 = \frac{C_{12} - C_{21}}{4q_0} = \frac{R_{21} - R_{12}}{4q_0} \tag{25-23}$$

SECTION 26

ANGULAR VELOCITY OF EYE FRAME RELATIVE TO HEAD FRAME IN TERMS OF RODRIGUES PARAMETERS

In this section, we derive an equation to express the angular velocity of the Eye frame relative to the Head frame with the components expressed in the Eye frame ($\mathbf{w}_{HE}^{E}$), in terms of the Rodrigues parameters ρ_1, ρ_2, and ρ_3, which we have discussed in Section 20.

In Section 25, we derived the equation for $\mathbf{w}_{HE}^{E}$ in terms of the quaternion elements q_0, q_1, q_2, and q_3. The equation is shown below:

$$\mathbf{w}_{HE}^{E} = 2\left[q_0\frac{d\mathbf{q}_v}{dt} - \frac{dq_0}{dt}\mathbf{q}_v\right] - 2\mathbf{q}_v \times \frac{d\mathbf{q}_v}{dt} \tag{26-1}$$

Here $\mathbf{q}_v$ denotes the vector part of a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. That is

$$\mathbf{q}_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \tag{26-2}$$

Our approach is to express each term on the right side of Equation (26-1) in terms of ρ_1, ρ_2, and ρ_3. Referring to Section 20:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} \dfrac{q_1}{q_0} \\ \dfrac{q_2}{q_0} \\ \dfrac{q_3}{q_0} \end{bmatrix} \tag{26-3}$$

It follows:

$$\mathbf{q}_v = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \rho_1 q_0 \\ \rho_2 q_0 \\ \rho_3 q_3 \end{bmatrix} \tag{26-4}$$

Using Equation (26-4):

$$\boldsymbol{\rho} \cdot \boldsymbol{\rho} = \rho_X^2 + \rho_Y^2 + \rho_Z^2 = \rho_1^2 + \rho_2^2 + \rho_3^2$$

$$= \frac{q_1^2 + q_2^2 + q_3^2}{q_0^2}$$

It follows:

$$1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho} = \frac{q_0^2 + q_1^2 + q_2^2 + q_3^2}{q_0^2}$$

$$= \frac{1}{q_0^2} \tag{26-5}$$

since $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ (see Section 19).

From Equation (26-5):

$$q_0 = \frac{1}{(1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{\frac{1}{2}}} = (1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{-\frac{1}{2}} \tag{26-6}$$

Now

$$\mathbf{q}_v = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

$$= \mathbf{i}\rho_1 q_0 + \mathbf{j}\rho_2 q_0 + \mathbf{k}\rho_3 q_0$$

$$= q_0(\mathbf{i}\rho_1 + \mathbf{j}\rho_2 + \mathbf{k}\rho_3)$$

$$= q_0\boldsymbol{\rho} \tag{26-7}$$

Substituting Equation (26-6) into Equation (26-7):

$$\mathbf{q}_v = \boldsymbol{\rho}(1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{-\frac{1}{2}} \tag{26-8}$$

Using the chain rule, the time–derivative of $\mathbf{q}_v$ becomes:

$$\dot{\mathbf{q}}_v = \frac{d\mathbf{q}_v}{dt} = \frac{d}{dt}\left[\boldsymbol{\rho}(1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{-\frac{1}{2}}\right]$$

$$= \frac{d\boldsymbol{\rho}}{dt}(1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{-\frac{1}{2}} + \boldsymbol{\rho}\frac{d}{dt}(1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{-\frac{1}{2}} \tag{26-9}$$

Referring to Equation (26-9):

$$\frac{d}{dt}(1 + \rho \cdot \rho)^{-\frac{1}{2}} = -\frac{1}{2}(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho + \rho \cdot \dot{\rho}\right)$$

$$= -\frac{1}{2}(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(2\dot{\rho} \cdot \rho\right)$$

$$= -(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho\right) \qquad (26\text{-}10)$$

Note that $\dot{\rho} \cdot \rho = \rho \cdot \dot{\rho}$ for dot product.

Using Equation (26-10) in Equation (26-9):

$$\dot{q}_v = \frac{d\rho}{dt}(1 + \rho \cdot \rho)^{-\frac{1}{2}} - \rho(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho\right) \qquad (26\text{-}11)$$

Now from Equation (26-6):

$$\dot{q}_0 = \frac{d}{dt}q_0 = \frac{d}{dt}(1 + \rho \cdot \rho)^{-\frac{1}{2}} \qquad (26\text{-}12)$$

$$= -\frac{1}{2}(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho + \rho \cdot \dot{\rho}\right)$$

$$= -(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho\right) \qquad (26\text{-}13)$$

Next, using Equation (26-6) and Equation (26-11):

$$q_0 \dot{q}_v = (1 + \rho \cdot \rho)^{-\frac{1}{2}}\left[\frac{d\rho}{dt}(1 + \rho \cdot \rho)^{-\frac{1}{2}} - \rho(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho\right)\right]$$

$$= (1 + \rho \cdot \rho)^{-1}\frac{d\rho}{dt} - \rho(1 + \rho \cdot \rho)^{-2}\left(\dot{\rho} \cdot \rho\right) \qquad (26\text{-}14)$$

Next, using Equation (26-13) and Equation (26-8):

$$\dot{q}_0 q_v = -(1 + \rho \cdot \rho)^{-\frac{3}{2}}\left(\dot{\rho} \cdot \rho\right)\left[\rho(1 + \rho \cdot \rho)^{-\frac{1}{2}}\right]$$

$$= -(1 + \rho \cdot \rho)^{-2}\rho\left(\dot{\rho} \cdot \rho\right) \qquad (26\text{-}15)$$

Finally, using Equation (26-8) and Equation (26-11):

$$\begin{aligned}\mathbf{q}_v \times \dot{\mathbf{q}}_v &= \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{1}{2}} \times \left[\dot{\boldsymbol{\rho}}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{1}{2}} - \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{3}{2}}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right)\right] \\ &= \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1} \times \dot{\boldsymbol{\rho}} - \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{1}{2}} \times \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{3}{2}}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right) \\ &= \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1} \times \dot{\boldsymbol{\rho}} - \boldsymbol{\rho} \times \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-2}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right) \\ &= \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1} \times \dot{\boldsymbol{\rho}} \\ &= \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1} \end{aligned} \tag{26-16}$$

since $\boldsymbol{\rho} \times \boldsymbol{\rho} = 0$, and since $\boldsymbol{\rho}\cdot\boldsymbol{\rho}$ is a scalar and therefore $(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1}$ is scalar as well.

Substituting Equation (26-6), Equation (26-11), Equation (26-15), and Equation (26-16) into Equation (26-1):

$$\mathbf{w} = 2\left[q_0\dot{\mathbf{q}}_v - \dot{q}_0\mathbf{q}_v - \mathbf{q}_v \times \dot{\mathbf{q}}_v\right]$$

$$= 2\left\{(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{1}{2}}\left[\dot{\boldsymbol{\rho}}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{1}{2}} - \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-\frac{3}{2}}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right)\right] + (1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-2}\ \boldsymbol{\rho}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right) - \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1}\right\}$$

$$= 2\left[(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1}\dot{\boldsymbol{\rho}} - \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-2}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right) + \boldsymbol{\rho}(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-2}\left(\dot{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\right) - \left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\right)(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1}\right]$$

$$= 2\left[(1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1}\dot{\boldsymbol{\rho}} - (1 + \boldsymbol{\rho}\cdot\boldsymbol{\rho})^{-1}\left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\right)\right] \tag{26-17}$$

It follows, factoring out $(1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho})^{-1}$:

$$\mathbf{w} = \mathbf{w}_{HE}^{E} = \frac{2}{1 + \boldsymbol{\rho} \cdot \boldsymbol{\rho}}\left[\frac{d\boldsymbol{\rho}}{dt} - \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{dt}\right] \qquad (26\text{-}18)$$

or in matrix form:

$$\mathbf{w} = \frac{2}{1 + \boldsymbol{\rho}^{T}\boldsymbol{\rho}} \left[1 - \rho^{*}\right] \frac{d\boldsymbol{\rho}}{dt} \qquad (26\text{-}19)$$

where

$$\rho^{*} = \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} \qquad (26\text{-}20)$$

$$\boldsymbol{\rho} \cdot \boldsymbol{\rho} = \boldsymbol{\rho}^{T}\boldsymbol{\rho} = \rho_X^2 + \rho_Y^2 + \rho_Z^2 \qquad (26\text{-}21)$$

and

$$\frac{d\rho}{dt} = \frac{d}{dt}\begin{bmatrix} \rho_X \\ \rho_Y \\ \rho_Z \end{bmatrix} \qquad (26\text{-}22)$$

SECTION 27

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APPENDIX A

DEMONSTRATION OF $(AB)^T = (AB)^{-1} = B^T A^T = B^{-1} A^{-1}$ (FOR 3 X 3 ORTHOGONAL MATRICES A AND B)

DEMONSTRATION OF $(AB)^T = (AB)^{-1} = B^T A^T = B^{-1} A^{-1}$ (FOR 3 X 3 ORTHOGONAL MATRICES A AND B)

This demonstration is straight forward, based on direct multiplications.

For Rotation Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{A-1}$$

$$A^T = A^{-1} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{A-2}$$

and

$$B^T = B^{-1} = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \tag{A-3}$$

since A and B matrices are othogonal.

It follows:

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \tag{A-4}$$

It follows by (1) exchanging rows and columns of Equation (A-4), (2) noting that the product of two Orthogonal Matrices is another Orthogonal Matrix, and (3) remembering that the transpose of an Orthogonal Matrix is equal to the inverse of the original matrix, that:

$$[AB]^T = [AB]^{-1} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \\ a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} & a_{21}b_{31} + a_{22}b_{23} + a_{23}b_{33} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix} \quad \text{(A-5)}$$

Now, from Equation (A-3) and Equation (A-2):

$$B^T A^T = B^{-1} A^{-1}$$

$$= \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} =$$

$$\begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} + b_{31}a_{13} & b_{11}a_{21} + b_{21}a_{22} + b_{31}a_{23} & b_{11}a_{31} + b_{21}a_{32} + b_{31}a_{33} \\ b_{12}a_{11} + b_{22}a_{12} + b_{32}a_{13} & b_{12}a_{21} + b_{22}a_{22} + b_{32}a_{23} & b_{12}a_{31} + b_{22}a_{32} + b_{32}a_{33} \\ b_{13}a_{11} + b_{23}a_{12} + b_{33}a_{13} & b_{13}a_{21} + b_{23}a_{22} + b_{33}a_{23} & b_{13}a_{31} + b_{23}a_{32} + b_{33}a_{33} \end{bmatrix} \quad \text{(A-6)}$$

Comparing Equation (A-5) and Equation (A-6), we conclude:

$$[AB]^T = [AB]^{-1} = B^T A^T = B^{-1} A^{-1} \quad \text{(A-7)}$$

Equation (A-7) is true only for the Orthogonal Matrices.

For Non–orthogonal Square Matrices, the formula for the transpose and the inverse are separate, and Equation (A-7) does not hold. That is,

$$[AB]^T = B^T A^T \quad \text{(A-8)}$$

$$[AB]^{-1} = B^{-1} A^{-1} \quad \text{(A-9)}$$

But, generally, $B^T A^T \neq B^{-1} A^{-1}$.

By extension:

$$[ABC]^T = [[AB]C]^T = C^T [AB]^T$$

$$= C^T B^T A^T \quad \text{(A-10)}$$

and

$$[ABC]^{-1} = [[AB]C]^{-1} = C^{-1}[AB]^{-1}$$

$$= C^{-1}B^{-1}A^{-1} \tag{A-11}$$

and

$$[ABC]^T = [ABC]^{-1} = C^TB^TA^T = C^{-1}B^{-1}A^{-1}.$$

Now denote vector **r** and $\mathbf{r}^T$ by:

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{r}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T = [x\ y\ z] \tag{A-12}$$

Then:

$$C\mathbf{r} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} C_{11}x + & C_{12}y + & C_{13}z \\ C_{21}x + & C_{22}y + & C_{23}z \\ C_{31}x + & C_{32}y + & C_{33}z \end{bmatrix} \tag{A-13}$$

It follows:

$$[C\mathbf{r}]^T = \begin{bmatrix} C_{11}x + C_{12}y + C_{13}z & C_{21}x + C_{22}y + C_{23}z & C_{31}x + C_{32}y + C_{33}z \end{bmatrix} \tag{A-14}$$

Now,

$$\mathbf{r}^TC^T = [x\ \ y\ \ z] \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11}x + C_{12}y + C_{13}z & C_{21}x + C_{22}y + C_{23}z & C_{31}x + C_{32}y + C_{33}z \end{bmatrix} \tag{A-15}$$

Comparing Equation (A-14) with Equation (A-15), we conclude

$$[\mathbf{Cr}]^T = \mathbf{r}^T C^T \tag{A-16}$$

Example: Application to Fick's System.

In Section 6, we found out that for the Fick's System of rotations,

$$\mathbf{r}^E = C_{F_2}^{F_3}\, C_{F_1}^{F_2}\, C_H^{F_1}\, \mathbf{r}^H \tag{A-17}$$

where

$$\mathbf{r}^E = \begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} \text{ and } \mathbf{r}^H = \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix}$$

Taking the transpose of Equation (A-17):

$$\left[\mathbf{r}^E\right]^T = \left[C_{F_2}^{F_3}\, C_{F_1}^{F_2}\, C_H^{F_1}\, \mathbf{r}^H\right]^T$$

$$= \left[\mathbf{r}^H\right]^T \left[\left[C_{F_2}^{F_3}\, C_{F_1}^{F_2}\right] C_H^{F_1}\right]^T$$

$$= \left[\mathbf{r}^H\right]^T = \left[\left[C_H^{F_1}\right]^T \left[C_{F_2}^{F_3}\, C_{F_1}^{F_2}\right]^T\right]$$

$$= \left[\mathbf{r}^H\right]^T = \left[C_H^{F_1}\right]^T \left[C_{F_1}^{F_2}\right]^T \left[C_{F_2}^{F_3}\right]^T \tag{A-18}$$

Now:

$$\left[\mathbf{r}^{E}\right]^{T} = \left[x_E \quad y_E \quad z_E\right]$$

$$\left[\mathbf{r}^{H}\right]^{T} = \left[x_H \quad y_H \quad z_H\right]$$

$$\left[C_{H}^{F_1}\right]^{T} = R_{H}^{F_1}$$

$$\left[C_{F_1}^{F_2}\right]^{T} = R_{F_1}^{F_2}$$

$$\left[C_{F_2}^{F_3}\right]^{T} = R_{F_2}^{F_3} \tag{A-19}$$

It follows from Equation (A-18) and Equation (A-19):

$$\left[x_E \; y_E \; z_E\right] = \left[x_H \; y_H \; z_H\right] R_{H}^{F_1} \, R_{F_1}^{F_2} \, R_{F_2}^{F_3} \tag{A-20}$$

in terms of R matrices.

Remember that the C matrix transforms the vector expressed in column vector form such as $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ from one frame to another, and the matrices representing subsequent rotations are multiplied from the **left** as shown in Equation (A-17). The R matrix transforms the vector form expressed in row vector such as $\mathbf{r}^T$ (x y z) from one frame to another, the matrices representing subsequent rotations are multiplied from the **right** as shown in Equation (A-20).

APPENDIX B

EQUATION OF MOTION FOR THE SEMICIRCULAR CANAL

EQUATION OF MOTION FOR THE SEMICIRCULAR CANAL[1]

The semicircular canal (SCC) is a rigid circular tube with a very small uniform cross–section. Each canal completes a hydrodynamic circuit through the utricles (which it shares in common with the other two canals on its side of the head) with negligible coupling with the other canals. The fluid flow within the canal is laminar because of the small, smooth bore of the canal. The canal has high viscous damping as indicated by the small Reynolds number of the flow. It is common to use lumped analysis and ignore the flow distribution within the canal. Thus, the fluid is considered to rotate as a ring relative to head.

The following analysis is based on angular rotation of the canal in a single plane. Consider a counterclockwise rotation (defined as the positive direction of rotation) of the canal and hence the head as described in Figure B-1. The damping torque (positive when acting counterclockwise) M_d on the fluid ring is assumed to be

$$M_d = -b\,\dot{\theta}(F/H) \tag{B-1}$$

where $\dot{\theta}\,(F/H)$ is the angular velocity of the fluid relative to the head and **b** is a positive proportionality constant. Note that $-b\,\dot{\theta}\,(F/H)$ is clockwise relative to the head in this case because of inertial reaction. The elastic restoring torque M_e on the fluid ring is assumed to be

$$M_e = -k\theta(F/H) \tag{B-2}$$

where θ (F/H) is the angular displacement of the fluid relative to the head and **k** is a positive proportionality constant.

1. Reprint from Bibliography 17 by Kee Soon Chun.

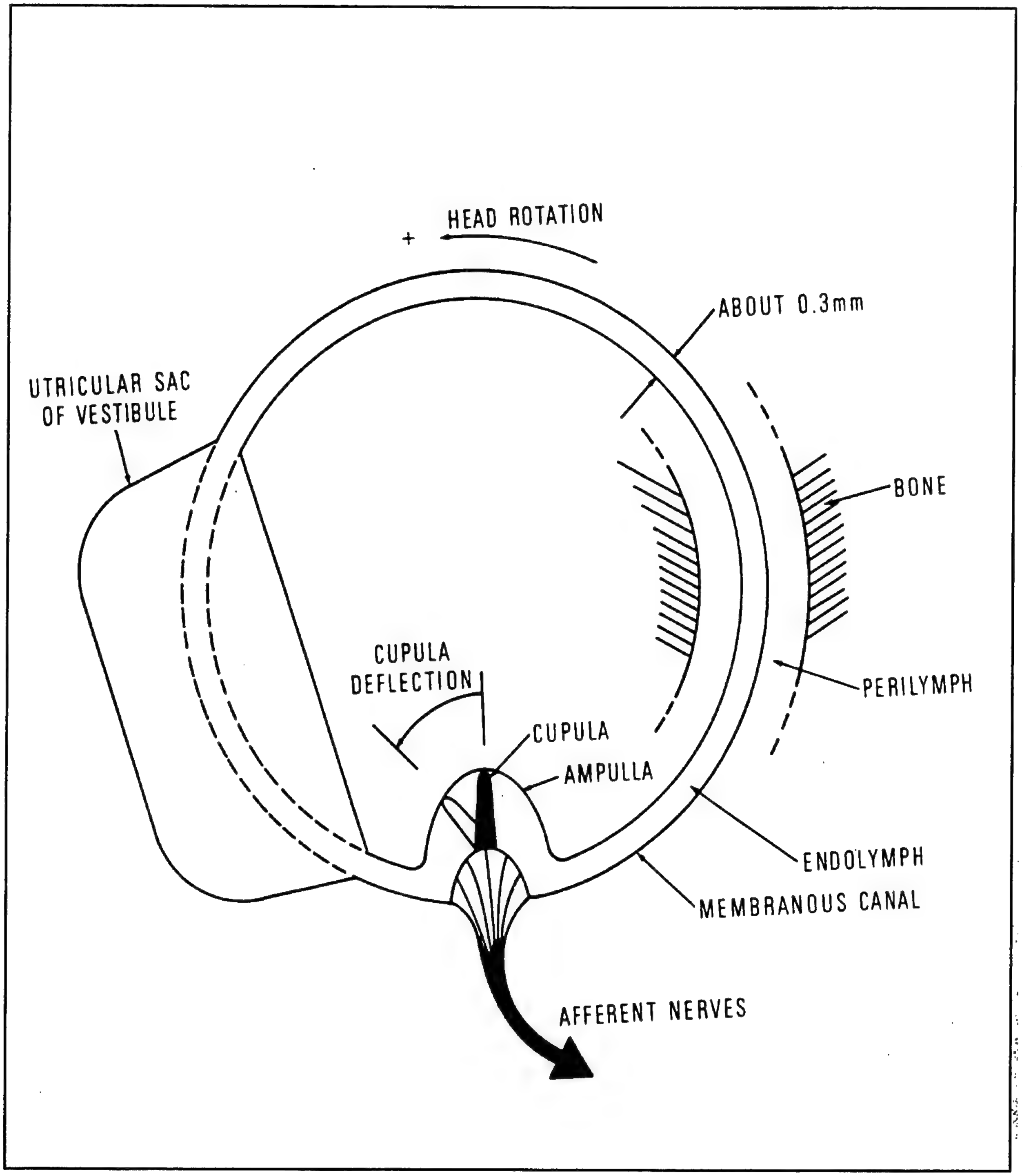

Figure B-1. The Semicircular Canal (diagrammatic)

By the rotational form of Newton's second law,

$$J\ddot{\theta}(F/I) = M_d + M_e = -b\dot{\theta}(F/H) - k\theta(F/H) \tag{B-3}$$

in which **J** is the fluid ring moment of inertia and $\ddot{\theta}(F/\mathbf{I})$ is the angular acceleration of the fluid relative to inertial space. By denoting the angular acceleration of the head relative to space by $\ddot{\theta}(H/\mathbf{I})$, it follows that:

$$\ddot{\theta}(F/I) = \ddot{\theta}(H/I) + \ddot{\theta}(F/H) \tag{B-4}$$

Assuming that the cupular deflection relative to the head $\theta(C/H)$ is proportional to the angular displacement of the head relative to fluid $\theta(H/F)$,

$$\theta(C/H) = a\theta(H/F) = -a\theta(F/H) \tag{B-5}$$

where **a** is a positive constant, which reflects the fact that the area of the canal is not equal to the area of the ampulla. Substituting Equation (B-4) and Equation (B-5) into Equation (B-3)

$$J\ddot{\theta}(C/H) + b\dot{\theta}(C/H) + k\theta(C/H) = aJ\ddot{\theta}(H/I) \tag{B-6}$$

Replacing cupular deflection $\theta(C/H)$ by ϕ and θ (H/I) by H in Equation (B-6),

$$J\ddot{\phi} + b\dot{\phi} + k\phi = aJ\ddot{H} \tag{B-7}$$

It is the same form as the differential equation for a torsion pendulum which describes the torsional vibrations of an elastic shaft with a circular rotor rigidly attached to it. The canal system is probably ten times more than critically damped.

Since the system represented by Equation (B-7) is very overdamped ($b^2 >> Jk$) and therefore $\frac{b}{k} > > \frac{J}{b}$, the transfer function (in Laplace transform notation) of the SCC may be approximated by:

$$\frac{\phi(s)}{\ddot{H}(s)} = \frac{aT_1T_2}{(T_1s + 1)(T_2s + 1)} \tag{B-8}$$

where

$$T_1 \simeq \frac{b}{k} \tag{B-9}$$

$$T_2 \simeq \frac{J}{b}$$

For humans, T_1 is thought to be about 10 ~ 16 sec and T_2 about 0.003 sec. For the frequency range below 5 Hz corresponding to normal daily activities, where $T_2 s < 1$, the transfer function may be further approximated as

$$\frac{\phi(s)}{\ddot{H}(s)} \simeq \frac{aT_1T_2}{T_1 s + 1} \tag{B-10}$$

Further, in the range above about 0.05 Hz, where $sT_1 > 1$, which includes most normal head movements,

$$\frac{\phi(s)}{\ddot{H}(s)} \simeq (aT_2)\,\frac{1}{s} \tag{B-11}$$

which is a pure integrator with gain (aT_2).

It follows that in this frequency range,

$$\phi\,(t) \propto \int \ddot{H}\,(t)\,dt \propto \dot{H}\,(t) \tag{B-12}$$

Thus, the output of the SCC is proportional to head velocity over the range of natural head movements, the role of SCC being that of an integrating accelerometer or velocity transducer. Indeed, the experimentally measured firing rate modulation of primary vestibular afferents is proportional to head velocity over this frequency range.

The gain constants of the internal neural signal processing elements are, for practical purposes, all indeterminable because the signals all consist of the firing rates of large populations of neurons, only a few of which can be observed at any one time. Only the final overall gain is important and interior gains may be adjusted arbitrarily so long as the total gain is kept correct. Thus, the gain of the transfer function Equation (B-10) of the SCC may be arbitrarily adjusted so that Equation (B-10) will behave like a pure integrator with a unity scale factor at midband frequencies. That is, T_1 is

relabeled the cupula long–time constant T_c and (aT_1T_2) is replaced by T_c. It follows from Equation (B-10) that:

$$\frac{\phi(s)}{\ddot{H}(s)} = \frac{T_c}{sT_c + 1} \tag{B-13}$$

which reduces to 1/s for $sT_c > 1$. The break frequency for $1/(sT_c + 1)$ is about 0.016 Hz with $T_c =$ 10 sec (for humans). Thus, for all frequencies of normal head rotation above 0.016 Hz, the canals integrate head acceleration and produce a signal proportional to head velocity.

APPENDIX C

EQUATION OF MOTION FOR THREE–DIMENSIONAL EYE ROTATIONS

EQUATION OF MOTION FOR THREE–DIMENSIONAL EYE ROTATIONS

We assume the Eye globe is a rigid sphere with a uniform density.

We use the standard Eye frame (Frame E) with x–axis forward, y–axis leftward, and z–axis upward with the origin at the center (which coincides with the center of mass) of the sphere.

We can easily see that each axis is the axis of symmetry in the sense that the rotation of the sphere about respective axis alter neither shape nor density distribution of the sphere.

This symmetry considerably simplifies the derivation of the equation of motion. Although the equation is well–established, it is derived here from scratch to familiarize readers with vector operations and with the application of the Coriolis Law.

By definition, the angular momentum (moment of momentum) $\mathbf{H}_C$ about C (the center of mass) of a point mass m_k located at point k with distance $\mathbf{R}_{Ck}$ from point C is (summing for all k's):

$$\mathbf{H}_C = \sum_k \left(\mathbf{R}_{Ck} \times m_k P_I \mathbf{R}_{Ck}\right) \tag{C-1}$$

where $P_I \mathbf{R}_{Ck} = \frac{d}{dt}\mathbf{R}_{Ck}$, with $\frac{d}{dt}$ ( ) performed in inertial space. By definition, the torque $\mathbf{M}_C$ about the center of mass C is

$$\mathbf{M}_C = \sum_k \mathbf{R}_{Ck} \times \mathbf{F}_k \tag{C-2}$$

where $\mathbf{F}_k$ is the external force exerted on the point k.

Differentiating Equation (C-1) with respect to time in inertial space (using chain rule):

$$\begin{aligned} P_I \mathbf{H}_C &= \sum_k \left(P_I \mathbf{R}_{Ck} \times m_k P_I \mathbf{R}_{Ck} + \mathbf{R}_{Ck} \times m_K P_I^2 \mathbf{R}_{Ck}\right) \\ &= \sum_k \left(\mathbf{R}_{Ck} \times m_k P_I^2 \mathbf{R}_{Ck}\right) \end{aligned} \tag{C-3}$$

because

$$P_I \mathbf{R}_{Ck} \times m_K P_I \mathbf{R}_{Ck} = m_K \left(P_I \mathbf{R}_{Ck} \times P_I \mathbf{R}_{Ck}\right) = 0. \tag{C-4}$$

since the mass m_k is a scalar.

Now (by vector addition),

$$\mathbf{R}_{Ik} = \mathbf{R}_{IC} + \mathbf{R}_{Ck} \tag{C-5}$$

where I is the origin of inertial frame, as shown in Figure C-1 below.

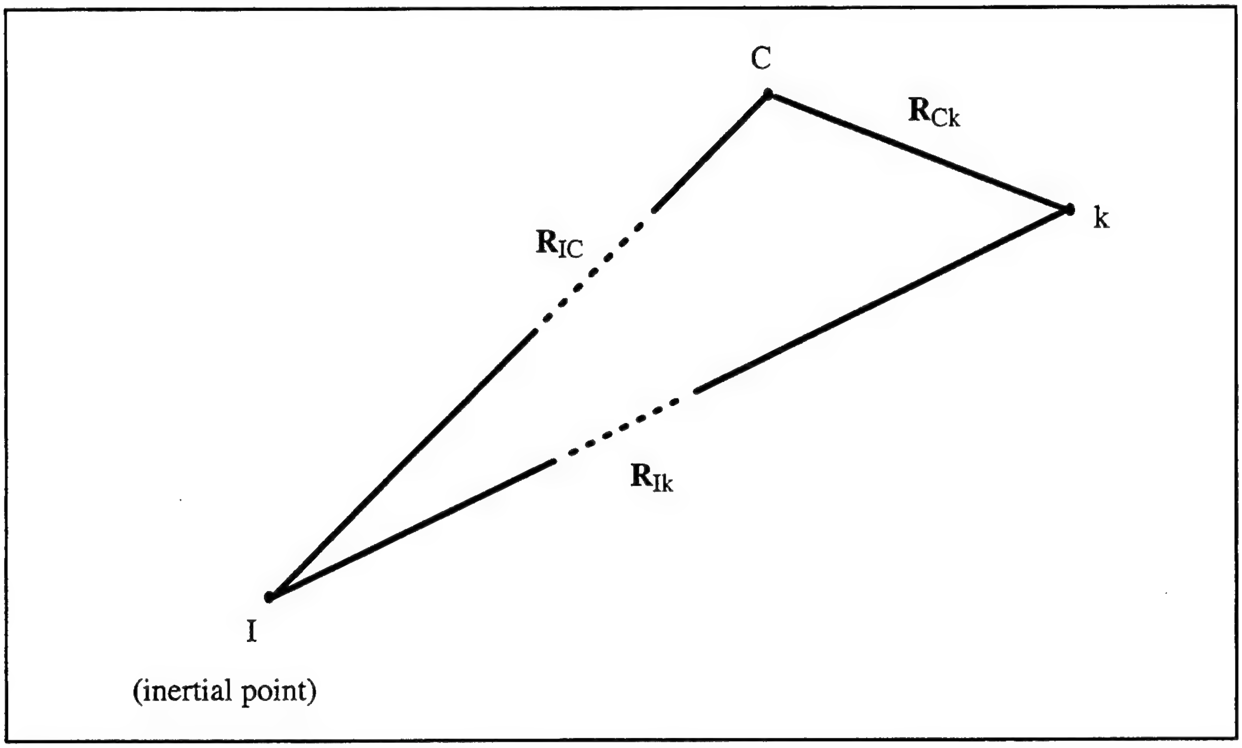

Figure C-1. I is Origin of Inertial Frame

From Equation (C-5):

$$\mathbf{R}_{Ck} = \mathbf{R}_{Ik} - \mathbf{R}_{IC} \tag{C-6}$$

Substituting Equation (C-6) into Equation (C-3):

$$P_I \mathbf{H}_C = \sum_k \left[\mathbf{R}_{Ck} \times m_k P_I^2 \left(\mathbf{R}_{Ik} - \mathbf{R}_{IC} \right) \right]$$

$$= \sum_k \left(\mathbf{R}_{Ck} \times m_K P_I^2 \mathbf{R}_{Ik} \right) - \sum_k \left(\mathbf{R}_{Ck} \times m_k P_I^2 \mathbf{R}_{IC} \right)$$

$$= \sum_k \left(\mathbf{R}_{Ck} \times \mathbf{F}_k \right)$$

$$- \left(\sum_k m_k \mathbf{R}_{Ck} \right) \times P^2 \mathbf{R}_{CK} \qquad \text{(C-7)}$$

because $m_k P_I^2 \mathbf{R}_{Ik} = \mathbf{F}_k$ by Newton's second law.

Now, $\sum_k m_k \mathbf{R}_{Ck} = 0$ by definition of the center of mass relative to the point C, which is also the origin of Frame E.

It follows from Equation (C-7) and Equation (C-2):

$$P_I \mathbf{H}_C = \sum_k \left(\mathbf{R}_{Ck} \times \mathbf{F}_k \right) = \mathbf{M}_C \qquad \text{(C-8)}$$

In other words, the applied torque about the center of mass is equal to the time rate of change of the angular momentum around the center of mass with respect to inertial space.

To determine $\mathbf{H}_C$ (the angular momentum about the center of mass) in terms of the coordinates x_k, y_k and z_k of $\mathbf{R}_k$, we start with Equation (C-1), which is repeated below:

$$\mathbf{H}_C = \sum_k \left(m_k \mathbf{R}_{Ck} \times P_I \mathbf{R}_{Ck} \right) \qquad \text{(C-9)}$$

According to the Coriolis Law, treating the Head frame as a fixed or reference frame and Eye frame as a rotating frame, we have:

$$P_H \mathbf{R}_{Ck} = P_E \mathbf{R}_{Ck} + \mathbf{w}_{HE} \times \mathbf{R}_{Ck} \qquad \text{(C-10)}$$

where

$$P_H \mathbf{R}_{Ck} = \frac{d}{dt}\mathbf{R}_{Ck} \quad \text{in Frame H}$$

$$P_E \mathbf{R}_{Ck} = \frac{d}{dt}\mathbf{R}_{Ck} \quad \text{in Frame E}$$

$\mathbf{w}_{HE}$ = angular velocity of Frame E relative to Frame H.

In Equation (C-10), $P_E \mathbf{R}_{Ck} = 0$ because $\mathbf{R}_{Ck}$ in eyeball is fixed. It follows from Equation (C-10):

$$P_H \mathbf{R}_{Ck} = \mathbf{w}_{HE} \times \mathbf{R}_{Ck} \tag{C-11}$$

Substituting Equation (C-11) into Equation (C-9):

$$\mathbf{H}_C = \sum_k \left[m_k \mathbf{R}_{Ck} \times \left(\mathbf{w}_{HE} \times \mathbf{R}_{Ck}\right)\right] \tag{C-12}$$

where we replaced $P_I \mathbf{R}_{Ck}$ by $P_H \mathbf{R}_{Ck}$ with good approximation.

From a standard mathematical table, we recall, for vectors $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$:

$$\mathbf{V}_1 \times \left(\mathbf{V}_2 \times \mathbf{V}_3\right) = \left(\mathbf{V}_1 \cdot \mathbf{V}_3\right)\mathbf{V}_2 - \left(\mathbf{V}_1 \cdot \mathbf{V}_2\right)\mathbf{V}_3 \tag{C-13}$$

Applying Equation (C-13) to Equation (C-12), identifying $\mathbf{V}_1$ with $m_k \mathbf{R}_{Ck}$, $\mathbf{V}_2$ with $\mathbf{w}_{HE}$, and $\mathbf{V}_3$ with $\mathbf{R}_{Ck}$:

$$\mathbf{H}_C = \sum_k \left(m_k \mathbf{R}_{Ck} \cdot \mathbf{R}_{Ck}\right)\mathbf{w}_{HE} - \sum_k \left(m_k \mathbf{R}_{Ck} \cdot \mathbf{w}_{HE}\right)\mathbf{R}_{Ck} \tag{C-14}$$

where

$$\mathbf{R}_{Ck} = x_k\mathbf{I} + y_k\mathbf{J} + z_k\mathbf{K} = \begin{bmatrix} x_k \\ y_k \\ z_K \end{bmatrix}^E = \mathbf{R}_{Ck}^E \tag{C-15}$$

$$\mathbf{w}_{HE} = w_X\mathbf{I} + w_Y\mathbf{J} + w_Z\mathbf{K} = \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix} = \mathbf{w}_{HE}^E \tag{C-16}$$

where the superscript E implied that the components are expressed in Frame E, or in Eye frame.

If coordinate axes are symmetric axes such as the axes of Frame E fixed to the Eye globe, the expression for $\mathbf{H}_C^E$ (with components expressed in Eye frame) expressed in terms of Equation (C-15) and Equation (C-16) becomes much simpler.

Substituting Equation (C-15) and Equation (C-16) in Equation (C-14), we have, after some algebra cancels out terms involving $x_k y_k$, $y_k z_k$, and $x_k z_k$:

$$\mathbf{H}_C^E = \begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix} = \begin{bmatrix} \left[\sum m_k\left(y_k^2 + z_k^2\right)\right] w_X \\ \left[\sum m_k\left(x_k^2 + z_k^2\right)\right] w_y \\ \left[\sum m_k\left(x_k^2 + y_k^2\right)\right] w_z \end{bmatrix}$$

$$= \begin{bmatrix} J_X\, w_X \\ J_Y\, w_Y \\ J_Z\, w_Z \end{bmatrix} \tag{C-17}$$

where the meaning of J_x, J_y, and J_z should be obvious from the first equation of Equation (C-17). It follows:

$$\mathbf{H}_C^E = \begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix}^E = \begin{bmatrix} \sum_k m_k\left(y_k^2 + z_k^2\right) & 0 & 0 \\ 0 & \sum_k m_k\left(x_k^2 + z_k^2\right) & 0 \\ 0 & 0 & \sum_k m_k\left(x_k^2 + y_k^2\right) \end{bmatrix} \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix} \tag{C-18}$$

or

$$\mathbf{H}_C^E = \begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix}^E = \begin{bmatrix} J_X & 0 & 0 \\ 0 & J_Y & 0 \\ 0 & 0 & J_Z \end{bmatrix} \begin{bmatrix} w_X \\ w_Y \\ w_Z \end{bmatrix} = \mathbf{J}\mathbf{w}_{HE}^E \tag{C-19}$$

where J represents the coefficient matrix on the right side of Equation (C-18). We use J instead of I to avoid confusion with Identity matrix.

Equation (C-17) to Equation (C-19) show that the direction of the angular momentum is the same as that of the angular velocity for Eye frame. However, in general, the angular momentum $\mathbf{H}_C$ and the angular velocity $\mathbf{w}$ of a rigid body are not in the same direction. By definition, the three axes of an orthogonal (mutually perpendicular) frame with this property are called **principal axes of inertia** or, briefly, **principal axes** of the body.

The symmetric axes, such as the axes of Eyeframe, are always principal axes. However, not all principal axes are symmetric axes. That is, there are principal axes that are not symmetric.

Now, returning to Equation (C-8), which is repeated below:

$$P_I \mathbf{H}_C = \mathbf{M}_C \tag{C-20}$$

In our study of eye rotations, we may regard the Head frame as inertial frame, with good approximation. So we may write Equation (C-20) as

$$P_H \mathbf{H}_C = \mathbf{M}_C \tag{C-21}$$

Applying the theorem of Coriolis between Head frame and Eye frame for $P_H H_C$ in Equation (C-21):

$$P_H \mathbf{H}_C = P_E \mathbf{H}_C + \mathbf{w}_{HE} \times \mathbf{H}_C = \mathbf{M}_C \tag{C-22}$$

In other words, the applied torque about the center of mass of eye produces and is equal to the rate of change of angular momentum with respect to the Eye frame plus the cross product of the angular velocity of the eye relative to the head and the angular momentum of the head about the center of mass.

Using notation given in Equation (C-16) and Equation (C-17):

$$\mathbf{w}_{HE} \times \mathbf{H}_C = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ w_X & w_Y & w_Z \\ H_X & H_Y & H_Z \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ w_X & w_Y & w_Z \\ J_X w_X & J_Y w_Y & J_Z w_Z \end{vmatrix}$$

$$= \begin{bmatrix} J_Z w_Y w_Z - J_Y w_Y w_Z \\ J_X w_X w_Z - J_Z w_X w_Z \\ J_Y w_X w_Y - J_X w_Y w_Y \end{bmatrix} \tag{C-23}$$

and

$$P_E \mathbf{H}_C = P_E \mathbf{H}_C^E = \frac{d}{dt}\begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix}$$

$$= \frac{d}{dt}\begin{bmatrix} J_X w_X \\ J_Y w_Y \\ J_Z w_Z \end{bmatrix} \tag{C-24}$$

$$= \frac{d}{dt}\begin{bmatrix} J_X \dot{\theta}_X \\ J_Y \dot{\theta}_Y \\ J_Z \dot{\theta}_Z \end{bmatrix} = \begin{bmatrix} J_X \ddot{\theta}_X \\ J_Y \ddot{\theta}_Y \\ J_Z \ddot{\theta}_Z \end{bmatrix} \tag{C-25}$$

Substituting Equation (C-23) and Equation (C-24) into Equation (C-22) and using $\mathbf{M}_C = \mathbf{M}_C^E = [M_X \; M_Y \; M_Z]^T$:

$$J_X \frac{d}{dt} w_X + (J_Z - J_Y)\, w_Y w_Z = M_X$$

$$J_y \frac{d}{dt} w_Y + (J_X - J_Z)\, w_X w_Z = M_Y \tag{C-26}$$

$$J_Z \frac{d}{dt} w_Z + (J_Y - J_X)\, w_X w_Y = M_Z$$

Equation (C-22) and Equation (C-26) apply to the body–fixed (eye–fixed in our case), principal axes of a rigid body (Eye frame axes that are symmetric in our case) with origin at the center of mass. In general, these equations represent highly nonlinear differential equations that are difficult to solve. The solution, when obtained, is the angular velocity of Eye globe with respect to Head expressed in Eye frame, and does not give directly the motion relative to Head axes.

For a special case of only an axis rotation, such as x–axis rotation, $w_X = w_Y = 0$. It follows from the first equation of (C-26),

$$J_X \frac{d}{dt} w_X = M_X \tag{C-27}$$

Denoting, $w_X \quad = dy \frac{d\theta_X}{dt} = \dot{\theta}_X$, we get from Equation (C-27) that

$$J_X \ddot{\theta}_X = M_X \tag{C-28}$$

which is commonly used in the one–dimensional analysis.

APPENDIX D

ACKNOWLEDGEMENTS

ACKNOWLEDGEMENTS

I gratefully acknowledge the U.S. Department of the Navy, and more specifically, to a significant contribution made by Mr. Frank Regan of NSWCDD, and fellow MIT alumni, who is the co–author of Bibliography 2 and the sole author of its first edition. For me, he is first class in spacecraft and missile dynamics.

I appreciate the effort made by Dr. Grace Peng of the Department of Neurology and Dr. Mehran Arnand of the Department of Otolaryngology of the Johns Hopkins School of Medicine for reviewing the manuscripts and making numerous comments and suggestions. I also received professional cooperation from the staff and students of the Vestibular Testing Laboratory of the Johns Hopkins School of Medicine, and sincere encouragement from its director, Dr. David Zee. It was nice of Professor David A. Robinson, my former dissertation advisor at Hopkins, to write the preface for the report after reviewing the manuscript. He also provided valuable comments.

I am thankful to Mr. Chris Kalivretenos, my Department Head at NSWCDD and to Dr. David Lando, my Division Head, and to Mr. Joseph F. Berenotto, my Branch Head, and to the K Department Administrative Assistant, Ms. Linda Loeffler, for their support. I would also like to take this opportunity to recognize the K53 Information Management Systems Group for their support and services, specifically, Mses. Patrecia Lipscomb and Jeanne Weaver. I also had several useful discussions with Doctors Jeffrey N. Blanton and James W. O'Toole, and Mr. Alan Rufty of NSWCDD, and appreciated their cooperation.

I gratefully express my profound respect and appreciation to my former teacher and mentor in the 1960's, the late Professor Walter Wrigley (co–author of Bibliography 1) of MIT Department of Aeronautics and Astronautics, who insisted on using in the classroom the step–by–step, confusion–free notations for rotational motions, that are adapted throughout this report.

I would also like to express my appreciation to a long–time family friend and a fellow graduate student at MIT, Jack Bortz. Seemingly countless discussions ensued with Jack in an attempt to understand Professor Wrigley's courses in those days. These discussions laid the foundation for whatever little I now know about spacecraft and missile dynamics. It was totally unexpected and came as a complete surprise a generation later while I am trying to combine two differential equations on Euler Vector Rate and Euler Angular Rate, to discover that the unified equation in matrix form for eye motion has turned out to be identical with the equation in vector form which he conceived and developed a generation ago for his thesis topic as a new concept in strapdown inertial navigation for the Apollo space program (Bibliograph 7).

Both of us feel we are lucky, near or at the end of our career mostly dedicated to the research on spacecraft and missile navigation and guidance, that what we did then and are doing now somehow may be contributing in a small way to the medical research and education, thus serving humanity as we talked about occasionally in those turbulent, yet fabulous '60s as graduate students.

Finally, I would like to recognize my wife, Byung–Oak. She deserves my appreciation for understanding and cooperating with the situation while I was writing this report. There were many weekends that I spent working on this report which could have been spent with family. I also thank my son, Walter, for initially motivating the writing of a book of this nature, when he was still a medical student.

Kee Soon Chun 田 基 順

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